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Spectral Petrov-Galerkin Methods for the Second Kind Volterra Type Integro-Differential Equations

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Abstract. This work is to provide general spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches for the second kind Volterra integro-differential equations. The Gauss-Legendre quadrature formula is used to approximate the integral operator and the inner product based on the Jacobi weight is implemented in the weak formulation in the numerical implementation. For some spectral and pseudo-spectral Jacobi-Petrov-Galerkin methods, a rigorous error analysis in both $L^2_{\omega^{\alpha,\beta}}$ and L^∞ norms is given provided that both the kernel function and the source function are sufficiently smooth. Numerical experiments validate the theoretical prediction.

AMS subject classifications: 65M10, 78A48

Key words: Volterra integro-differential equation, spectral Jacobi-Petrov-Galerkin, pseudo-spectral Jacobi-Petrov-Galerkin, spectral convergence.

1. Introduction

This paper is concerned with the following second-kind Volterra integro-differential equation with initial condition, i.e.,

$$\begin{cases} u'(x) + \displaystyle\int_{-1}^{x} k(x,s)u(s)ds = g(x), \qquad x \in [-1,1], \\ u(-1) = 0, \end{cases} \tag{1.1}$$

where the kernel function $k(x,s)$ and the source function $g(x)$ are given smooth functions, $u(x)$ is the unknown function.

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Actually any second-kind Volterra integro-differential equation with smooth kernel and initial condition can be transformed into (1.1) by a simple linear transformation used in [12]. As a result, our approach can be generalized to the second-kind Volterra integro-differential equations with initial condition defined in any interval, where the kernel is smooth. We will consider the case that the solutions of (1.1) are sufficiently smooth. Consequently it is natural to implement very high-order numerical methods such as spectral methods for the solutions of (1.1). It is known that there are many numerical approaches for solving (1.1), such as collocation methods, finite element methods, see, e.g., [1] and references therein. Nevertheless, few works touched the spectral approximations to (1.1). In [9], Chebyshev spectral methods are proposed to solve nonlinear Volterra-Hammerstein integral equations. Then Chebyshev spectral methods are investigated in [10] for the first kind Fredholm integral equations under multiple-precision arithmetic. Nevertheless, no theoretical results are provided to justify the high accuracy numerically obtained. Some efforts are made to implement the spectral methods to solve the second-kind Volterra integral equations. In [14], a spectral method is suggested, but spectral accuracy is not observed for most of the computations. Tang and Xu in [12] develop a novel spectral Legendre-collocation method. Actually this is the first spectral approach for which the spectral accuracy can be justified both theoretically and numerically. Inspired by the work in [12], Chen and Tang [4] implement the spectral Chebyshev-collocation method to solve the second kind Volterra integral equation with weakly singular kernel $(t-s)^{-\frac{1}{2}}k(t,s)$, where $k(t,s)$ is a smooth function. Then they [5] extend the approach in [4] to the second kind Volterra integral equation with more general weakly singular kernel $(t-s)^{\alpha}k(t,s)$, where $-1<\alpha\le 0$ and $k(t,s)$ is a smooth function. The spectral accuracy of these approaches is verified both theoretically and numerically in [4] and [5]. Xie and Tang [7] develop spectral and pseudo-spectral Galerkin methods based on the general Jacobi weight to solve the second-kind Volterra integral equation. They give a rigorous proof of the spectral convergence in $L^2_{\omega^{\alpha,\beta}}$ and L^∞ norms. Actually, the success of the spectral method for the second-kind Volterra integral equations is the main motivation for our work in the second-kind integro-differential equations.

Unlike the standard spectral and pseudo-spectral Galerkin methods, the spectral and pseudo-spectral Petrov-Galerkin methods allow the trial and test function spaces to be different. Lin et.al, in [8] introduce the Petrov-Galerkin finite element (PGFE) method for Volterra integro-differential equations. It is proved that the PGFE solution u_h and its derivative u_h' have optimal convergence rates $\mathcal{O}(h^{m+1})$ and $\mathcal{O}(h^m)$ in L^∞ norm, respectively. After using some postprocessing techniques, the convergence rate of u_h reaches $\mathcal{O}(h^{2m})$ at the nodes of the mesh. Tang [13] discusses the collocation method to solve the first-order Volterra integro-differential equation with a singular kernel function $(t-s)^{-\alpha}k(t,s,u(s))(0<\alpha<1)$. For grading exponents $r>\frac{m}{2-\alpha}$ of the graded mesh, the collocation solution has the convergence rate $\mathcal{O}(N^{-m})$ in L^∞ norm. Besides, Brunner, *et.al*, in [2] present the $hp-$discontinuous Galerkin method for Volterra integro-differential equations with singular kernels. It is proved both theoretically and numerically that the DG solution based on geometrically graded meshes has the exponential convergence rate in L^2 and L^∞ norms. Inspired by these works, we will show that both spectral and pseudo-