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The Partition of Unity Method for High-Order Finite Volume Schemes Using Radial Basis Functions Reconstruction

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Abstract. This paper introduces the use of partition of unity method for the development of a high order finite volume discretization scheme on unstructured grids for solving diffusion models based on partial differential equations. The unknown function and its gradient can be accurately reconstructed using high order optimal recovery based on radial basis functions. The methodology proposed is applied to the noise removal problem in functional surfaces and images. Numerical results demonstrate the effectiveness of the new numerical approach and provide experimental order of convergence.

AMS subject classifications: 65N99, 68U10, 65F22
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1. Introduction

Evolutionary nonlinear partial differential equations (PDEs) are nowadays a well assessed tool in image, surface processing and computer vision. The main image and surface processing applications involving PDE models are nonlinear filtering, edge/feature detection, image deblurring and enhancement, restoration, inpainting, segmentation, shape extraction and analysis, motion analysis, see, e.g., [7, 17, 22]. The time discretization of the PDE models is usually obtained by explicit or implicit methods while the space discretization is provided by finite element (FEM), finite difference (FD) or finite volumes (FV) schemes covering the domain by suitable grids. In image processing structured grids are simple to handle, while in surface processing block structured or unstructured grids are of common usage. FEM and FV methods have been used successfully to solve problems of image multi-scale analysis. In particular, since FV schemes are directly based on the integral form of the conservation law and because the numerical flux is based on the physics of

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nonlinear wave propagation, the FV schemes are thus able to cope with discontinuities in the solution, and thus are particularly suitable to deal with image or surfaces domains [8].

The goal of our work is to construct a new numerical method, that we name "partition of unity high order finite volume" (PUHOFV) method, which solves PDEs with, at least theoretically, arbitrary high order of accuracy in space on unstructured meshes complex domains. In particular, we investigate certain PDE diffusion models in the context of image and surface denoising. The main ingredient of the proposed PUHOFV scheme is a new reconstruction strategy based on partition of unity and RBF optimal recovery, which makes use of simple regularization techniques allowing for a robust computation on strongly unstructured grids.

The partition of unity framework is a powerful technique to approximate a global function blending together the local approximations; it is able to model discontinuities and singularities through local enrichment. Initially introduced as a general FEM method in computational mechanics, see, e.g., [1,14], the partition of unity approach has become popular also in the computer graphics community within shape reconstruction setting, [27]. In this work, the partition of unity method is used to provide an easy to use and accurate approximation framework for finite volume schemes on unstructured grids.

The finite volume method is based on the discretization of the solution domain into a set of non-overlapping finite volumes and thereafter, the integral representation of the underlying conservation laws are approximated over these volumes using some appropriate numerical strategy [26]. In the cell-centered approach the computed quantities are stored on each cell and the centroid values of the dependent variable play an important role in the interpolation methods required to reconstruct fluxes. However, a high accuracy reconstruction should be combined with an accurate integral approximation method [25].

Traditionally, the midpoint rule has been the favored method to approximate the line integrals involving these fluxes. However, it achieves second order accuracy only when the flux evaluations are sufficiently accurate. We will consider Gaussian quadrature methods for integration in order to obtain an arbitrary desired precision.

The basic FV schemes offer a piecewise constant solution representation in space; they are very robust but provide only a first order accuracy, thus requiring to refine the grid tremendously to obtain the desired accuracy, with a consequent overhead in time and memory requirements. This leads naturally into the theory of optimal recovery. Since Barth and Frederickson's pioneering work [2], a number of researchers have studied high order FV methods using unstructured meshes. The reason for developing very high order schemes is that they permit a good resolution of physical phenomena even on very coarse grids and that they exhibit only very little numerical dissipation, which is important when performing simulations in large domains for long times. Furthermore, grid refinement becomes much more efficient using high order schemes since numerical errors decrease faster compared to the case when the same grid refinement is applied using a low order scheme.

The key aspect of a high order accurate FV solver is a high order reconstruction by means of piecewise smooth functions from the cell average values. To this aim, a variety of techniques have been explored to reconstruct the fluxes to at least second order accuracy