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A Fourth-Order Modified Method for the Cauchy Problem of the Modified Helmholtz Equation

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Abstract. This paper is concerned with the Cauchy problem for the modified Helmholtz equation in an infinite strip domain $0 < x \leq 1$, $y \in \mathbb{R}$. The Cauchy data at $x = 0$ is given and the solution is then sought for the interval $0 < x \leq 1$. This problem is highly ill-posed and the solution (if it exists) does not depend continuously on the given data. In this paper, we propose a fourth-order modified method to solve the Cauchy problem. Convergence estimates are presented under the suitable choices of regularization parameters and the *a priori* assumption on the bounds of the exact solution. Numerical implementation is considered and the numerical examples show that our proposed method is effective and stable.

AMS subject classifications: 65M32

Key words: Cauchy problem for the modified Helmholtz equation, ill-posed problem, fourth-order modified method.

1. Introduction

The modified Helmholtz equation arises in many areas, especially in practical physical applications, such as implicit marching schemes for the heat equation, Debye-Huckel theory and the linearization of the Poisson-Boltzmann equation, see, e.g., [1,2,8,11,12]. The direct problems, i.e., the Dirichlet, Neumann or mixed boundary value problems for the modified Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of boundary or at some interior points of the concerned domain. This is called an inverse problem. The Cauchy problem (a function that satisfies a partial differential equation and the Dirichlet and Neumann boundary conditions which are given on a part of the boundary) for the modified Helmholtz equation is an inverse problem and is severely ill-posed, which means the solution does not depend continuously on the given Cauchy data (the given Dirichlet and Neumann data) and any small change in the given data may cause large change to the solution [7,22]. Several numerical

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methods have been proposed to solve this problem, such as the alternating iterative algorithm based on the Landweber method in conjunction with the boundary element method (BEM) [13], the conjugate gradient method with the BEM [14], the method of fundamental solution [10,15,24]. In [9], the boundary knot method was applied to solve the Cauchy problem for the inhomogeneous Helmholtz equation.

In this paper, we will consider the Cauchy problem of the modified Helmholtz equation in an infinite strip domain $0 < x < 1$, $y \in \mathbb{R}$ as follows

$$-\triangle u(x,y) + k^2 u(x,y) = 0, \quad x \in (0,1), \quad y \in \mathbb{R}, \tag{1.1}$$

$$u(0,y) = \varphi(y), \quad y \in \mathbb{R}, \tag{1.2}$$

$$u_x(0,y) = 0, \qquad y \in \mathbb{R}. \tag{1.3}$$

The problem (1.1)-(1.3) is ill-posed, see Section 2 below and [6,7]. One kind of regularization methods is to cut off high frequencies, see, e.g., [16,20].

In this paper, we use a modified method to solve the Cauchy problem for the modified Helmholtz equation (1.1)-(1.3). That is, we modify the original equation (1.1) to give the following fourth-order equation,

$$-\Delta u(x,y) + k^2 u(x,y) + \mu^2 u_{xxyy}(x,y) = 0. \tag{1.4}$$

The basic idea originated from Weber's paper [23] and then Elden's paper [3] in which the authors used a similar method to solve a standard inverse heat conduction problem. This kind of method has been used to solve a wide range of ill-posed problems. For example, the Cauchy problem for the Laplace equation, the backward heat conduction problem and the sideways heat equation, see, e.g., [4,17–19,21].

The paper is organized as follows. In Section 2, we consider the ill-posedness of the proposed problem and propose a fourth-order modified method. In Section 3, the convergence estimates under the suitable choices of regularization parameters are established. The numerical implementation is discussed in Section 4. Some conclusions are given in Section 5.

2. Ill-posedness and a modified method

In this section, we will analyze the ill-posedness of the Cauchy problem (1.1)-(1.3) in the frequency space and give a modified method for obtaining a stable approximate solution.

Define the Fourier transform of a function as follows,

$$\hat{\varphi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(y) \exp(-i\xi y) dy.$$

To solve the Cauchy problem (1.1)-(1.3), we take the Fourier transform with respect to variable y to Eq. (1.1) and boundary conditions (1.2)-(1.3). Then the Cauchy problem