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NUMERICAL ANALYSIS OF MODULAR VMS METHODS WITH NONLINEAR EDDY VISCOSITY FOR THE NAVIER-STOKES EQUATIONS

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Abstract. This paper presents the stabilities for both two modular, projection-based variational multiscale (VMS) methods and the error analysis for only first one for the incompressible Naiver-Stokes equations, expanding the analysis in [39] to include nonlinear eddy viscosities. In VMS methods, the influence of the unresolved scales onto the resolved small scales is modeled by a Smagorinsky-type turbulent viscosity acting only on the marginally resolved scales. Different realization of VMS models arise through different models of fluctuations. We analyze a method of inducing a VMS treatment of turbulence in an existing NSE discretization through an additional, uncoupled projection step. We prove stability, identifying the VMS model and numerical dissipation and give an error estimate. Numerical tests are given that confirm and illustrate the theoretical estimates. One method uses a fully nonlinear step inducing the VMS discretization. The second induces a nonlinear eddy viscosity model with a linear solve of much less cost.

Key words. Navier-Stokes equations, eddy viscosity, projection-based VMS method, uncoupled approach.

1. Introduction

Variational multiscale (VMS) methods have proven to be an important approach to the numerical simulation of turbulent flows (see Section 1.1 for its genesis and some recent work). VMS methods are efficient, clever and simple realization of the idea of introducing eddy viscosity locally in scale space only on the marginally resolved scales. They add dissipation to mimic the loss of energy in the marginally resolved scales caused by breakdown of eddies to unresolved scales through a term of the form:

$$\Big(\nu_T(\mathbf{u}^h)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}(I-P_H)\mathbf{v}^h\Big), \tag{1}$$

where $\mathbb{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$ is the velocity deformation tensor (symmetric part of the gradient), P_H is an elliptic projection onto the well-resolved velocities on a given mesh (so $(I-P_H)\mathbf{u}^h$ represents the marginally resolved velocity scales).

The success of VMS methods leads naturally to the question of how to introduce them into legacy codes and other multi-physics codes so large as to discourage abandoning a method or a model that is already implemented to reprogram another one. In [39], this question was addressed: a VMS method can be induced into a black box (even laminar) flow simulation by adding a modular projection step, uncoupled from the (possibly black box) flow code. Although the numerical tests were quite general, the mathematical/numerical analysis in [39] in support of modular VMS methods was for constant eddy viscosity parametrizations $\nu_T(\cdot)$. In this report we continue the mathematical support for modular VMS methods in two ways. First we expand the analysis of [39] to include the fully nonlinear, eddy

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viscosity case of the (ideal) "small-small" Smagorinsky model (1) above for which

$$\nu_T(\mathbf{u}^h) = (C_s\delta)^2 \big|\mathbb{D}(I - P_H)\mathbf{u}^h\big|_F, \tag{2}$$

The motivation of the Smagorinsky model is to replicate the decay of energy due to breakdown of eddies from resolved to unresolved scales in the energy cascade, [6, 7, 8, 15, 21, 22, 29, 31, 32, 34, 43, 45]. The ideal case of (2) has the most complete mathematical theory due to the strong monotonicity on the marginally resolved s-cales of (1) with (2), Section 3. Unfortunately, the choice (2) also increases the cost of implementing a VMS method in a modular Step 2. We therefore consider methods (i) whose realization is as close as possible to the ideal small-small S-magorinsky model, (ii) for which a complete and rigorous mathematical foundation can be given, and (iii) whose implementation is comparable in cost and complexity to the linear case of $\nu_T \equiv$constant. These issues lead to our second, related method with eddy viscosity term:

$$\Big(A_e\big(\nu_T(\mathbf{u}^h)\big)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}(I-P_H)\mathbf{v}^h\Big) = \Big(A_e\big(\nu_T(\mathbf{u}^h)\big)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}\mathbf{v}^h\Big) \tag{3}$$

where $A_e\big(\nu_T(\cdot)\big)$ is an element average over the elements (e.g. triangles in 2d) which define the well-resolved scales, see Definition 4.1. Because the eddy viscosity coefficient $A_e\big(\nu_T(\cdot)\big)$ is now elementwise constant, simplifications arise in the modular Step 2 below which enforces the VMS turbulence model. The restriction to elementwise constant eddy viscosities originates in the works of Lube and Roehe [44] on full (or monolithic) VMS methods.

To introduce the idea of [39] developed herein, suppose the Navier-Stokes equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}) + \nu A\mathbf{u} = \mathbf{f}(t). \tag{4}$$

Let Π denote a postprocessing operator. The method we extends and analyzes, adds one uncoupled postprocessing step to a given method (we select the commonly used Crank-Nicolson time discretization for Step 1 for specificity): given $\mathbf{u}^n \cong \mathbf{u}(t^n)$, compute $\mathbf{u}^{n+1}$ by

Step 1: Compute $\mathbf{w}^{n+1}$ via:

$$\frac{\mathbf{w}^{n+1} - \mathbf{u}^n}{\triangle t} + N\Big(\frac{\mathbf{w}^{n+1} + \mathbf{u}^n}{2}\Big) + \nu A \frac{\mathbf{w}^{n+1} + \mathbf{u}^n}{2} = \mathbf{f}^{n+\frac{1}{2}}. \tag{5}$$

Step 2: Postprocess $\mathbf{w}^{n+1}$ to obtain $\mathbf{u}^{n+1}$:

$$\mathbf{u}^{n+1} = \Pi\mathbf{w}^{n+1}. \tag{6}$$

Both steps can be done by uncoupled modules. Eliminating Step 2 gives:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} + N\Big(\frac{\mathbf{w}^{n+1} + \mathbf{u}^n}{2}\Big) + \nu A \frac{\mathbf{w}^{n+1} + \mathbf{u}^n}{2} + \frac{1}{\triangle t}(\mathbf{w}^{n+1} - \Pi\mathbf{w}^{n+1}) = \mathbf{f}^{n+\frac{1}{2}}, \tag{7}$$

where $\mathbf{f}^{n+\frac{1}{2}} = (\mathbf{f}^{n+1} + \mathbf{f}^n)/2$. We define the operator Π in Step 2, following [39] so that the extra term is exactly a nonlinear Smagorinsky model acting on small resolved scales.

$$\frac{1}{\triangle t}(\mathbf{w}^{n+1} - \mathbf{u}^{n+1}, \mathbf{v}_h) = (Smagorinsky\ Model, \mathbf{v}_h). \tag{8}$$

We consider herein two algorithmic realizations of (8). The first method analyzed is a full Smagorinsky model. Let P_{L^H} denote an L^2 projection onto a space of "well resolved" deformations, see Definition 1.2 for a precise formulation in Section 1.2.