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STABLE COMPUTING WITH AN ENHANCED PHYSICS BASED SCHEME FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. We study extensions of the energy and helicity preserving scheme for the 3D Navier-Stokes equations, developed in [23], to a more general class of problems. The scheme is studied together with stabilizations of grad-div type in order to mitigate the effect of the Bernoulli pressure error on the velocity error. We prove stability, convergence, discuss conservation properties, and present numerical experiments that demonstrate the advantages of the scheme.

Key Words. Finite element method, Discrete helicity conservation, Grad-div stabilization

1. Introduction

This paper extends the methodology of the enhanced-physics based scheme for the 3D Navier-Stokes equations (NSE) proposed in [23] (defined in Section 2) from its original derivation for space-periodic problems to a more general class of problems. This scheme is referred to as *enhanced-physics* because it is the only scheme that conserves *both* discrete energy and discrete helicity for the full 3D NSE. The key ingredient for the dual conservation scheme is using the rotational form of the nonlinearity with a projected vorticity, which allows the discrete nonlinearity to preserve both of the quantities. Since the (continuous) NSE nonlinearity conserves both energy and helicity, and jointly cascades them from the large scales through the inertial range to small viscosity dominated scales [3, 5], if the discrete nonlinearity does not also conserve energy and helicity it will introduce numerical error into the cascade, and bring into question the physical relevance of computed approximations.

It is a widely held belief in computational fluid dynamics (CFD) that the more *physically correct* a numerical scheme is, the more accurate its predictions will be, especially over long time intervals. In systems of conservation laws for fluids there is typically a second integral invariant in addition to energy, and its accurate treatment in a numerical scheme generally produces more accurate simulations than do schemes that do not specifically conserve this quantity. Beginning with Arakawa's energy and enstrophy conserving scheme for the 2D NSE [1] and related extensions [8], to energy and potential enstrophy schemes pioneered by Arakawa and Lamb, and Navon, [2, 19, 20], and most recently to an energy and helicity conserving scheme for 3D axisymmetric flow by J.-G. Liu and W. Wang [16], enhanced physics

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based schemes have provided more accurate simulations, especially over longer time intervals.

The fundamental challenge in extending the scheme of [23] to non-periodic problems is to avoid the large errors often present when the rotational form of the nonlinearity and the Bernoulli pressure is used. In the usual a priori error analysis for the velocity approximation for the NSE, a consequence that the discrete divergence free velocity is not exactly divergence free, is a pressure error contribution

$$\frac{C}{\nu} \inf_{q_h \in Q_h} \|p - q_h\| \tag{1.1}$$

where $\nu = 1/$Reynolds number denotes the kinematic viscosity [9, 15]. For problems whose pressure gradients are small this term is often negligible. However, using the rotational form of the NSE, and introducing the Bernoulli pressure $p + \frac{1}{2}|u|^2$ can bring prominence to this term, since the gradient of the Bernoulli pressure may be large due to boundary layers in the velocity field.

Following recent work in [14, 17, 4], a natural way to mitigate the pressure's error influence on the velocity approximation is to introduce grad-div stabilization. As we show, this reduces the effect of the Bernoulli pressure error. In the interest of physical fidelity, we also introduce a modified grad-div stabilization having the same effect on the error, but with less impact on the energy balance. Computational results show a slight improvement when this altered stabilization is used instead of usual grad-div stabilization.

This paper is arranged as follows. Section 2 presents mathematical preliminaries and notation, and defines the scheme studied in the remainder of the article. Section 3 is a study of stability and conservation laws for the scheme, and Section 4 presents a rigorous convergence analysis. Section 5 shows a numerical example which clearly illustrates the advantage of the scheme. Concluding remarks are given in Section 6.

2. Mathematical Preliminaries

We assume that Ω denotes a polyhedral domain in $\mathbb{R}^3$. The $L^2(\Omega)$ norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the seminorm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $v(x,t)$ defined on the entire time interval $[0,T]$, we define $(1 \le m < \infty)$

$$\|v\|_{\infty,k} := \operatorname*{ess\,sup}_{[0,T]} \|v(t,\cdot)\|_k \,, \text{ and } \|v\|_{m,k} := \left(\int_0^T \|v(t,\cdot)\|_k^m \, dt\right)^{1/m} .$$

For the analysis in this paper, we assume no slip (i.e. homogeneous Dirichlet) boundary conditions for velocity and therefore use as our velocity and pressure spaces

$$X := (H_0^1(\Omega))^d, \; Q := L_0^2(\Omega) \,,$$

where Q is denoting the *mean zero* subspace of $L^2(\Omega)$.

We use as the norm on X the H^1 seminorm which, because of the boundary condition, is a norm, i.e. for $v \in X$, $\|v\|_X := \|\nabla v\|$. We denote the dual space of X by $X^\star$, with the norm $\|\cdot\|_\star$. The space of divergence free functions is defined by

$$V := \{v \in X : \; (\nabla \cdot v, q) = 0 \;\; \forall q \in Q\} \,.$$