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A PRIORI ERROR ESTIMATES FOR SEMIDISCRETE FINITE ELEMENT APPROXIMATIONS TO EQUATIONS OF MOTION ARISING IN OLDROYD FLUIDS OF ORDER ONE

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Abstract. In this paper, a semidiscrete finite element Galerkin method for the equations of motion arising in the 2D Oldroyd model of viscoelastic fluids of order one with the forcing term independent of time or in L^∞ in time, is analyzed. A step-by-step proof of the estimate in the Dirichlet norm for the velocity term which is uniform in time is derived for the nonsmooth initial data. Further, new regularity results are obtained which reflect the behavior of solutions as $t \to 0$ and $t \to \infty$. Optimal $L^\infty(\mathbf{L}^2)$ error estimates for the velocity which is of order $O(t^{-1/2}h^2)$ and for the pressure term which is of order $O(t^{-1/2}h)$ are proved for the spatial discretization using conforming elements, when the initial data is divergence free and in H_0^1. Moreover, compared to the results available in the literature even for the Navier-Stokes equations, the singular behavior of the pressure estimate as $t \to 0$, is improved by an order $1/2$, from t^{-1} to $t^{-1/2}$, when conforming elements are used. Finally, under the uniqueness condition, error estimates are shown to be uniform in time.

Key Words. Viscoelastic fluids, Oldroyd fluid of order one, uniform *a priori* bound in Dirichlet norm, uniform in time and optimal error estimates, nonsmooth initial data.

1. Introduction

In this paper, we consider semi-discrete Galerkin approximations to the following system of equations of motion arising in the Oldroyd fluids (see, J. G. Oldroyd ([23])) of order one:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - \int_0^t \beta(t-\tau)\Delta\mathbf{u}(x,\tau)\,d\tau + \nabla p = \mathbf{f}(x,t), \tag{1.1}$$

with $x \in \Omega,\ t > 0$ and incompressibility condition

$$\nabla\cdot\mathbf{u} = 0,\ x\in\Omega,\ t>0, \tag{1.2}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \text{ in } \Omega,\ \mathbf{u} = 0, \text{ on } \partial\Omega,\ t \ge 0. \tag{1.3}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, $\mu = 2\kappa\lambda^{-1} > 0$ and the kernel $\beta(t) = \gamma\exp(-\delta t)$, where $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$ and $\delta = \lambda^{-1} > 0$. We note that ν is the kinematic coefficient of viscosity. λ is the relaxation time, and is characterized by the fact that after instantaneous cessation of motion, the stresses

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in the fluid do not vanish instantaneously, but die out like $exp(-\lambda^{-1}t)$. Moreover, the velocities of the flow, after instantaneous removal of the stresses, die out like $exp(-\kappa^{-1}t)$, where κ is the retardation time. For further details of the physical background and its mathematical modeling, we refer to [14], [23] and [24].

There is considerable amount of literature devoted to Oldroyd model by Russian mathematicians such as A.P.Oskolkov, Kotsiolis, Karzeeva and Sobolevskii etc, see [24, 1, 8, 15, 18] and references, therein. Based on the analysis of Ladyzhenskaya [19] for the Navier-Stokes equations, Oskolkov [24] has proved existence of a unique 'almost' classical solution in finite time interval $[0,T]$ for the $2D$ problem (1.1)-(1.3). In the proof, the constant appeared in *a priori* bounds depends exponentially on T and therefore, it is not possible to extend the results for large time. Subsequently, Agranovich and Sobolevskii [1] have extended the analysis of Oskolkov and have derived global existence of solutions for all $t \geq 0$ when $f \in L^2(\mathbf{L}^2)$ with smallness conditions on data for $3D$ problem. The solvability on the semi-axis $t \geq 0$, for the problem (1.1)-(1.3), is discussed in [18, 8] when $\mathbf{f}, \mathbf{f}_t \in L_\infty(\mathbb{R}^+; \mathbf{L}^2(\Omega))$ in [18] and $\mathbf{f}, \mathbf{f}_t \in S^2(\mathbb{R}^+; \mathbf{L}^2(\Omega))$ in [8], where S^2 is a subspace of L^2_{loc}. We observe that results in [18, 8] hold true only for finite time ($T < \infty$), that is, for $\mathbf{f}, \mathbf{f}_t \in L_\infty(0, T; \mathbf{L}^2(\Omega))$, with estimate depending on T, but there seems to have some difficulties in extending these results for all $t \geq 0$, when $\mathbf{f}, \mathbf{f}_t \in L_\infty(\mathbb{R}^+; \mathbf{L}^2(\Omega))$. For example, in [18], it is difficult to derive the estimate (20) from (17) on page 2780 by applying integral version of the Gronwall's Lemma and the estimate (12). Unfortunately, this is further carried over to subsequent articles, see Theorem 2 of [8]. In the context of dynamical system generated by Oldroyd model when $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, see, [15] and [17], it is not quite clear that the conclusion of Theorem 1.2 of [15] for $s = 1$ or Theorem 1 of [17] for $\ell = 1$ holds true. In fact a more careful observation in both these articles demands an estimate of $\int_0^1 \|\boldsymbol{\phi}\|^2_{E_1}$, which is difficult to establish prior to this result. In the context of 2D Navier-Stokes equations, a standard tool for deriving uniform Dirichlet norm for the velocity term is to apply uniform Gronwall's Lemma. Due to the presence of the integral term in (1.1), it is difficult to apply uniform Gronwall's Lemma (see Remark 3.3(i)). To be more precise, on the right-hand side of (17) on page 2780 of [18], the estimate of $\int_0^t \sum_{l=1}^L \beta_l \|\tilde{\Delta}\mathbf{u}_l\|^2_{2,\Omega_t} d\tau$ is not available from (12) (for notations, see [18]), which is crucial in applying uniform Gronwall's Lemma. This is exactly a similar problem faced in article [15] and [17].

In [30], Sobolevskii has examined the behavior of the solution as $t \to \infty$ under some stabilization conditions like positivity of the first eigenvalue of a selfadjoint spectral problem introduced therein and Hölder continuity of the function $\Phi = e^{\delta_0 t}(\mathbf{f}(x,t) - \mathbf{f}_\infty(x))$, where $\mathbf{f}_\infty = \overline{\lim}_{t\to\infty} \mathbf{f}$ and $\delta_0 > 0$, using energy arguments and positivity of the integral operator, see also Kotsiolis and Oskolkov [16]. Recently, He *et al.* [10] have proved similar results under milder conditions on $\mathbf{f}$ and weaker regularity assumptions on the initial data $\mathbf{u}_0$. In fact, in their analysis, they have shown both the power and exponential convergence of the solutions to a steady state solution, when $\Phi \in L^\infty(\mathbf{L}^2)$ only.

For the numerical approximations to the problem (1.1)-(1.3), we refer to Akhmatov and Oskolkov [2], Cannon *et al.* [5], He *et al.* [11] and Pani *et al.* [28]. In [2], stable finite difference schemes are discussed without any discussion on convergence. Cannon *et al.* [5] have proposed a modified nonlinear Galerkin scheme for (1.1)-(1.3) with periodic boundary condition using a spectral Galerkin method and have discussed convergence analysis while keeping time variable continuous. In [11], local optimal error estimates for the velocity in $L^\infty(\mathbf{H}^1)$-norm and the pressure in $L^\infty(L^2)$-norm are established. Moreover, these estimates are shown to be uniform