Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 3, pp. 309-320

DOI: 10.4208/aamm.12-m12110
June 2013

Numerical Integration over Pyramids

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Received 26 August 2012; Accepted (in revised version) 19 November 2012

Available online 30 April 2013

Abstract. Pyramidal elements are often used to connect tetrahedral and hexahedral elements in the finite element method. In this paper we derive three new higher order numerical cubature formulae for pyramidal elements.

AMS subject classifications: 65D30, 65D32, 65N30

Key words: Reference pyramidal element, nonlinear systems of algebraic equations, Bramble-Hilbert lemma, triangular, tetrahedral and pyramidal numbers.

1 Introduction

Let

$$K=\big\{(x,y,z)\in\mathbb{R}^3\big|\ |x|,|y|\le 1-z,\ \ 0\le z\le 1\big\}$$

be the reference pyramidal element. For a continuous function f of K we shall look for the numerical integration formulae

$$\int_K f(x,y,z)dxdydz\approx\sum_{m=0}^{n}\omega_m f(A_m), \tag{1.1}$$

where weights $\omega_m\in\mathbb{R}^1$ and at the same time positions of nodes $A_m\in K$ are appropriately chosen so that Eq. (1.1) is exact for all polynomials of the highest possible degree.

Pyramidal elements are natural and useful for making face-to-face connections between tetrahedral and hexahedral elements in approximating the solutions of three-dimensional initial and boundary value problems by the finite element method (see

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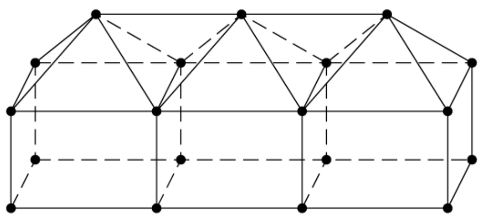

Figure 1:

Fig. 1). This often happens when one part of the solution domain is decomposed into hexahedra and the other into tetrahedra (usually near a curved boundary).

In 1997 it was independently observed in [8] and [15] that a conforming finite element method cannot be achieved with polynomial shape functions on pyramids. This surprising statement was later exactly proved in [12] Liu et al., namely, that there is no continuously differentiable function on the pyramid K that would be linear on its four triangular faces and bilinear, but not linear, on its rectangular base. Therefore, in [12] and [13] three symmetric composite finite elements with 5, 13, and 14 degrees of freedom were introduced. Their piecewise polynomial shape functions on each pyramid yield a conforming finite element space. Another way is to apply a nonconforming finite element method (see, e.g., [2]), where finite element functions are, in general, discontinuous on interior faces in a partition involving pyramidal elements. In this case we have to integrate polynomials and other smooth functions over pyramids to calculate the stiffness matrix and the corresponding right-hand side (the load vector). For instance, the famous discontinuous Galerkin method belongs to the class of nonconforming methods.

Numerical integration formulae on tetrahedra, prisms or hexahedra are very well studied in the literature (see, e.g., [5,7,11]). However, up to the authors's knowledge, there are only a few papers dealing with numerical integration on pyramids. For instance, nothing about this topic is mentioned in the encyclopedia [6]. Some special cubature formulae on pyramids are treated in [9,10,14]. In [1] a bilinear surjective vector mapping F from the unit cube to the reference pyramid K is proposed. The whole upper face of the cube is mapped onto the upper vertex of K. Numerical integration on K is then derived from the standard Gaussian formulae on the unit cube by means of the mapping F. For instance, the numerical cubature formula that is exact for all cubic polynomials has 8 nodal points inside the cube. Their images are inside of K, but the four upper integration points are somewhat unnaturally accumulated near the top vertex $(0,0,1)$. Moreover, the corresponding numerical cubature formula (see Eq. (4.2)) is not exact for all cubic polynomials. When solving nonlinear three-dimensional problems, numerical cubature formulae usually cannot be avoided, since the entries of the stiffness matrix and/or the right-hand side cannot be evaluated analytically.

In Sections 2 and 3, we derive new numerical cubature formulae which are exact for all quadratic and cubic polynomials and they have only 5 and 6 integration points, respectively. Our formulae are different from those presented in [1,9,10,14]. Section