INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 1, Pages 125–155



# SPACE–TIME ADAPTATION FOR PURELY DIFFUSIVE PROBLEMS IN AN ANISOTROPIC FRAMEWORK

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(Communicated by Paul Houston)

*This paper is dedicated to Mario*

**Abstract.** The main goal of this work is the proposal of an efficient space-time adaptive procedure for a cGdG approximation of an unsteady diffusion problem. We derive a suitable a posteriori error estimator where the contribution of the spatial and of the temporal discretization is kept distinct. In particular our interest is addressed to phenomena characterized by temporal multiscale as well as strong spatial directionalities. On the one hand we devise a sound criterion to update the time step, able to follow the evolution of the problem under investigation. On the other hand we exploit an anisotropic triangular adapted grid. The reliability and the efficiency of the proposed error estimator are assessed numerically.

**Key Words.** space-time adaptation, anisotropic meshes, heat equation, space-time finite elements.

## 1. Introduction and motivations

We are interested in devising an effective procedure for selecting both the time step and the grid size to approximate unsteady diffusion problems. Such models are employed to describe applications of interest in, e.g., heat flow problems, hydrogeology, particle diffusion phenomena, etc.

We propose an adaptation algorithm based on both an adaptive choice of the temporal step and an anisotropic mesh adaptation strategy. Indeed the applications we have in mind often exhibit both temporal multiscale phenomena and spatial heterogeneities. The first issue calls for a time step, fitting the evolution of the phenomenon at hand; the second occurs in the presence of lower dimensional features of the domain, of the problem data or of the solution.

The proposed adaptation procedure relies on a theoretical tool, i.e., an a posteriori error estimator, driving the automatic choice of the spatial and temporal steps. In more detail, the key point is to identify, in the error estimator, a space and a time contribution. This is aligned with other works such as, e.g., R. Verfürth [51], J.M. Cascón, L. Ferragut & M.I. Asensio [9], D. Meidner & B. Vexler [33], M. Schmich & B. Vexler [44]. The additional value of our work is the possibility of dealing also with problems characterized by strong spatial directional features by means of an economic mesh from the computational viewpoint. It is in fact well known

Received by the editors May 10, 2008, in the revised form, February 6, 2009.

2000 *Mathematics Subject Classification.* 35K20, 65M50, 65M60.

This research was supported by COFIN 2006 “Metodi Numerici Avanzati per il Calcolo Scientifico”.

that, by better orienting the mesh elements according to the main features of the solution, it is possible to maximize the solution accuracy for a fixed number of elements, rather than reduce the number of degrees of freedom for a fixed solution accuracy (see, e.g., M.J. Castro–Díaz, F. Hecht, B. Mohammadi & O. Pironneau [10], J. Dompierre, M.-G. Vallet, Y. Bourgault, M. Fortin & W.G. Habashi [18], T. Apel [2], D.L. Darmofal & D.A. Venditti [15], L. Formaggia & S. Perotto [22], K.G. Siebert [47], E.H. Georgoulis, E. Hall & P. Houston [24]).
In addition, the proposed adaptive algorithm carries out the spatial adaptation via an optimization procedure, in contrast to the more familiar mark-refine approach (Z. Chen & J. Feng [11], [9, 33, 44]). On the other hand the temporal step is adaptively updated driven by the time residual contribution rather than by a standard fixed-ratio reduction, as, e.g., in A. Schmidt & K.G. Siebert [45], [11, 9, 33], M. Picasso [41]. Alternative methods to drive the space-time adaptation are provided, for instance, in B. Cockburn & C.-W. Shu [14], R. Hartmann & P. Houston [25], D. Kröner & M. Ohlberger [30], J.J. Sudirham, J.J.W. van der Vegt & R.M.J. van Damme [48].

To approximate the problem under investigation we adopt a discretization scheme appropriate to manage space and time in parallel, i.e., based on space-time finite elements. In particular we choose finite elements continuous in space but discontinuous in time, i.e., the so-called cGdG scheme (K. Eriksson, C. Johnson & V. Thomée [21], K. Eriksson, D. Estep, P. Hansbo & C. Johnson [19]).
In compliance with G. Akrivis, C. Makridakis & R. Nochetto [1], M. Picasso [40], K. Eriksson & C. Johnson [20] we are interested in controlling a suitable global norm of the space-time error discretization, in our case through an anisotropic residual-based like a posteriori estimator. Even though the discretization framework that we provide is quite general and suited for a generic cGdG method, the anisotropic a posteriori analysis is confined to the cG(1)dG(0) scheme due to the particular anisotropic setting employed ([22], L. Formaggia & S. Perotto [23]).
The main difference with respect to a corresponding isotropic error estimator (R. Verfürth [50], [51]) is the presence of an anisotropic contribution weighting the standard residuals, and depending on the actual error. To preserve the anisotropic information without giving up the computability of the estimator, we resort to a suitable recovery approach: as for the space, we employ the well-performing Zienkiewicz-Zhu gradient recovery procedure (O.C. Zienkiewicz & J.Z. Zhu [52, 53, 54]); the time recovery is based on the idea in [33].
The choice of resorting to the Zienkiewicz-Zhu methodology can be attributed to various factors: the method is rather independent of the problem, of the governing equations and of most details of the finite element formulation (except for the finite element space); it is cheap to compute, easy to implement and works very well in practice (see, e.g., J.C. Bruch [6], K.L. Lawrence & R.V. Nambiar [31], T.P. Pawlak, M.J. Wheeler & S.M. Yunus [39]). Moreover we have already exploited such an approach also in an anisotropic framework with successful results (see, e.g., G. Maisano, S. Micheletti, S. Perotto & C.L. Bottasso [32], S. Micheletti & S. Perotto [34], L. Dedè, S. Micheletti & S. Perotto [17]).

As far as we know, the only paper available in the current literature dealing with an anisotropic management of triangular grids in a time-dependent framework is [41]. However in this last work the time adaptation issue is completely skipped. The author simply introduces the successive halving of the time step so that the time discretization contribution becomes negligible with respect to the spatial one. Moreover in this case a standard backward Euler scheme is adopted to discretize