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Correlation Functions, Universal Ratios and Goldstone Mode Singularities in n-Vector Models

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Abstract. Correlation functions in the $\mathcal{O}(n)$ models below the critical temperature are considered. Based on Monte Carlo (MC) data, we confirm the fact stated earlier by Engels and Vogt, that the transverse two-plane correlation function of the $\mathcal{O}(4)$ model for lattice sizes about $L=120$ and small external fields h is very well described by a Gaussian approximation. However, we show that fits of not lower quality are provided by certain non-Gaussian approximation. We have also tested larger lattice sizes, up to $L=512$. The Fourier-transformed transverse and longitudinal two-point correlation functions have Goldstone mode singularities in the thermodynamic limit at $k\to 0$ and $h=+0$, i.e., $G_\perp(\mathbf{k})\simeq ak^{-\lambda_\perp}$ and $G_\parallel(\mathbf{k})\simeq bk^{-\lambda_\parallel}$, respectively. Here a and b are the amplitudes, $k=|\mathbf{k}|$ is the magnitude of the wave vector $\mathbf{k}$. The exponents $\lambda_\perp$, $\lambda_\parallel$ and the ratio bM^2/a^2, where M is the spontaneous magnetization, are universal according to the GFD (grouping of Feynman diagrams) approach. Here we find that the universality follows also from the standard (Gaussian) theory, yielding $bM^2/a^2=(n-1)/16$. Our MC estimates of this ratio are 0.06 ± 0.01 for $n=2$, 0.17 ± 0.01 for $n=4$ and 0.498 ± 0.010 for $n=10$. According to these and our earlier MC results, the asymptotic behavior and Goldstone mode singularities are not exactly described by the standard theory. This is expected from the GFD theory. We have found appropriate analytic approximations for $G_\perp(\mathbf{k})$ and $G_\parallel(\mathbf{k})$, well fitting the simulation data for small k. We have used them to test the Patashinski-Pokrovski relation and have found that it holds approximately.

AMS subject classifications: 65C05, 82B20, 82B80

Key words: n-component vector models, correlation functions, Monte Carlo simulation, Goldstone mode singularities.

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1 Introduction

The n-component vector-spin models (called also n-vector models or $\mathcal{O}(n)$ models), have attracted significant interest in recent decades as the models, where the so-called Goldstone mode singularities are observed. The Hamiltonian of the n-vector model $\mathcal{H}$ is given by

$$\frac{\mathcal{H}}{T} = -\beta\Big(\sum_{\langle ij \rangle} \mathbf{s}_i \mathbf{s}_j + \sum_i \mathbf{h}\mathbf{s}_i\Big), \tag{1.1}$$

where T is temperature, $\mathbf{s}_i \equiv \mathbf{s}(\mathbf{x}_i)$ is the n-component vector of unit length, i.e., the spin variable of the i-th lattice site with coordinate $\mathbf{x}_i$, β is the coupling constant, and $\mathbf{h}$ is the external field. The summation takes place over all nearest neighbors in the lattice. Periodic boundary conditions are considered here.

In the thermodynamic limit below the critical temperature (at $\beta > \beta_c$), the magnetization $M(h)$ (where $h = |\mathbf{h}|$), the Fourier-transformed transverse ($G_\perp(\mathbf{k})$) and longitudinal ($G_\parallel(\mathbf{k})$) two-point correlation functions exhibit Goldstone mode power-law singularities:

$$M(h) - M(+0) \propto h^\rho \qquad \text{at } h \to 0, \tag{1.2a}$$

$$G_\perp(\mathbf{k}) = a k^{-\lambda_\perp} \qquad \text{at } h = +0 \quad \text{and} \quad k \to 0, \tag{1.2b}$$

$$G_\parallel(\mathbf{k}) = b k^{-\lambda_\parallel} \qquad \text{at } h = +0 \quad \text{and} \quad k \to 0, \tag{1.2c}$$

with certain exponents ρ, $\lambda_\perp$, $\lambda_\parallel$ and the amplitudes a, b of the Fourier-transformed two-point correlation functions.

In a series of theoretical works (e.g., [1–11]), it has been claimed that the exponents in (1.2a)-(1.2c) are exactly $\rho = 1/2$ at $d = 3$, $\lambda_\perp = 2$ and $\lambda_\parallel = 4 - d$, where d is the spatial dimensionality $2 < d < 4$. These theoretical approaches are further referred here as the standard theory. Several MC simulations have been performed earlier [12–15] to verify the compatibility of MC data with standard-theoretical expressions, where the exponents are fixed. In recent years, we have performed a series of accurate MC simulations [16–19] for remarkably larger lattices than previously were available, with an aim to evaluate the exponents in (1.2a)-(1.2c). Some deviations from the standard-theoretical values have been observed, in agreement with an alternative theoretical approach, known as the GFD (grouping of Feynman diagrams) theory [20], where the relations $d/2 < \lambda_\perp < 2$, $\lambda_\parallel = 2\lambda_\perp - d$ and $\rho = (d/\lambda_\perp) - 1$ have been found for $2 < d < 4$.

In the GFD theory, the perturbation theory is reorganized in such a way that all Feynman diagrams are summed up into certain skeleton diagrams, where the true correlation function (instead of the Gaussian one) corresponds to the coupling lines. Further grouping and resummation of all these skeleton diagrams allows us to perform a qualitative analysis without cutting the perturbation series. Possible reasons for discrepancies between the GFD theory and standard perturbative treatments are discussed in [18]. This discussion is mainly devoted to the critical point singularities, but the same arguments refer also to the Goldstone mode singularities. There are some difficulties in the GFD