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Cohomology in 3D Magneto-Quasistatics Modeling

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Abstract. Electromagnetic modeling provides an interesting context to present a link between physical phenomena and homology and cohomology theories. Over the past twenty-five years, a considerable effort has been invested by the computational electromagnetics community to develop fast and general techniques for defining potentials. When magneto-quasi-static discrete formulations based on magnetic scalar potential are employed in problems which involve conductive regions with holes, *cuts* are needed to make the boundary value problem well defined. While an intimate connection with homology theory has been quickly recognized, heuristic definitions of cuts are surprisingly still dominant in the literature.
The aim of this paper is first to survey several definitions of cuts together with their shortcomings. Then, cuts are defined as generators of the first cohomology group over integers of a finite CW-complex. This provably general definition has also the virtue of providing an automatic, general and efficient algorithm for the computation of cuts. Some counter-examples show that heuristic definitions of cuts should be abandoned. The use of cohomology theory is not an option but the invaluable tool expressly needed to solve this problem.

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Key words: Algebraic topology, (co)homology, computational electromagnetics, cuts.

1 Introduction

There is a remarkable interest in the efficient numerical solution of large-scale three-dimensional electromagnetic problems by Computer-Aided Engineering (CAE) softwares which enables a rapid and cheap design of practical devices together with their optimization.

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Electromagnetic phenomena are governed by Maxwell's laws [1] and constitutive relations of materials. This paper focuses on the numerical solution of magneto-quasi-static Boundary Value Problems (BVP)—also called eddy-current problems—which neglect the *displacement current* in the Ampère-Maxwell's equation [1–3]. This well-studied class has quite a big number of industrial applications such as non-destructive testing, electromagnetic breaking, metal separation in waste, induction heating, metal detectors, medical imaging and hyperthermia cancer treatment.

The range of CAE applications is sometimes bounded by the high computational cost needed to obtain the solution, hence state-of-the-art numerical methods are usually sought. Recently, the Discrete Geometric Approach (DGA) gained popularity, becoming an attractive method to solve BVP arising in various physical theories, see for example [4–14]. The DGA bears strong similarities to compatible or mimetic discretizations [15, 16], discrete exterior calculus [17] and finite element exterior calculus [18–20]. All these methods present some pedagogical advantages with respect to the standard widely used Finite Element Method (FEM).

First of all, the topological nature of Maxwell's equations and the geometric structure behind them allows to reformulate the mathematical description of electromagnetism directly in algebraic form. Such a reformulation can be formalized in an elegant way by using algebraic topology [5, 6, 8, 9, 16, 18–20]. Taking advantage of this formalism, as illustrated in Section 3, physical variables are modeled as cochains and Maxwell's laws are enforced by means of the coboundary operator. Information about the metric and the physical properties of the materials is encoded in the constitutive relations, that are modeled as discrete counterparts of the Hodge star operator [8, 11, 16, 20, 21] usually called *constitutive matrices* [13]. By combining Maxwell's with constitutive matrices, an algebraic system of equations is directly obtained, yielding to a simple, accurate and efficient numerical technique. The difference of the DGA with respect to similar methods lies in the computation of the constitutive matrices, which in the DGA framework is based on a closed-form geometric construction. For a computational domain discretized by using a geometric realization of a polyhedral cell complex, one may use the techniques described in [22, 23] and references therein, without losing the symmetry, positive-definiteness and consistency of the constitutive matrices which guarantee the convergence of the method. Hence, we consider the most general situation of dealing with a polyhedral cell complex.

Our purpose is not to present the widely known DGA or similar discretizations, but to use it as a working framework. This choice does not limit the generality of the results, since the standard Finite Element Method (FEM) and the Finite Differences (FD) can be easily reinterpreted in the DGA framework as in [8,10,11,16,18–20,24,25]. Consequently our results can be extended, without any modification, to the corresponding widely used FEM formulation.

The paper is focused on a particular application of algebraic topology, namely the definition of potentials for the efficient numerical solution of eddy-currents Boundary Value Problems (BVP). Electromagnetic potentials are auxiliary quantities frequently used to enforce some of the Maxwell's laws implicitly. There are two families of formulations for