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On Arbitrary-Lagrangian-Eulerian One-Step WENO Schemes for Stiff Hyperbolic Balance Laws

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Abstract. In this article we present a new family of high order accurate Arbitrary Lagrangian-Eulerian one-step WENO finite volume schemes for the solution of stiff hyperbolic balance laws. High order accuracy in space is obtained with a standard WENO reconstruction algorithm and high order in time is obtained using the local space-time discontinuous Galerkin method recently proposed in [20]. In the Lagrangian framework considered here, the local space-time DG predictor is based on a weak formulation of the governing PDE on a moving space-time element. For the space-time basis and test functions we use Lagrange interpolation polynomials defined by tensor-product Gauss-Legendre quadrature points. The moving space-time elements are mapped to a reference element using an isoparametric approach, i.e. the space-time mapping is defined by the same basis functions as the weak solution of the PDE. We show some computational examples in one space-dimension for non-stiff and for stiff balance laws, in particular for the Euler equations of compressible gas dynamics, for the resistive relativistic MHD equations, and for the relativistic radiation hydrodynamics equations. Numerical convergence results are presented for the stiff case up to sixth order of accuracy in space and time and for the non-stiff case up to eighth order of accuracy in space and time.

AMS subject classifications: 65Mxx, 35Lxx

Key words: Arbitrary Lagrangian-Eulerian, finite volume scheme, moving mesh, high order WENO reconstruction, local space-time DG predictor, moving isoparametric space-time elements, stiff relaxation source terms, Euler equations, resistive relativistic MHD equations, relativistic radiation hydrodynamics.

1 Introduction

We present a new class of high order one-step Arbitrary Lagrangian-Eulerian (ALE) finite volume schemes for stiff hyperbolic balance laws. While the mesh is fixed in an

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Eulerian description, in Lagrangian type schemes the computational mesh moves with the local fluid velocity. That means that material interfaces are moving together with the mesh and thus one can precisely identify their location. In the recent past, a lot of work has been carried out to develop Lagrangian methods. Some algorithms are developed starting directly from the conservative quantities such as mass, momentum and total energy [47, 59] while another class starts from the nonconservative form of the governing equations [4, 6, 70]. In any discrete scheme one has to decide where to place the degrees of freedom of each physical variable. The existing Lagrangian schemes in literature can be generally separated into two main classes: 1) staggered mesh methods, where the velocity is defined at the cell interfaces while the other physical variables are located at the cell center and 2) cell-centered methods, where all variables are defined at the cell center.

In [52] Munz presented several different Godunov-type finite volume schemes for Lagrangian gas dynamics and, in particular, he was the first to introduce a Roe linearization for Lagrangian gas dynamics. It was found that the Lagrangian Roe linearization actually does *not* coincide with the Eulerian one [52]. The resulting maximum signal speeds of this Roe linearization have subsequently been used to construct robust HLL-type Riemann solvers in Lagrangian coordinates. Carré et al. [7] describe a cell-centered Godunov scheme for Lagrangian gas dynamics on general multi-dimensional unstructured meshes. Their finite volume solver is node based and compatible with the mesh displacement. In [17], Després and Mazeran propose a way of writing the equations of gas dynamics in Lagrangian coordinates in two dimensions as a weakly hyperbolic system of conservation laws. The system contains both the physical and the geometrical part. Based on the symmetrization of the formulation of the physical part, the authors design a finite volume scheme for the discretization of Lagrangian gas dynamics on moving meshes. In [39], Jua and Zhang present a high-order Lagrangian Runge-Kutta DG scheme for the discretization of two-dimensional compressible gas dynamics. The scheme uses a fully Lagrangian form of the gas dynamics equations and employs a new HWENO-type reconstruction algorithm as limiter to control the spurious oscillations, maintaining the compactness of RKDG methods. The time marching for the semi discrete schemes for the physical and geometrical variables is implemented by a classical TVD Runge-Kutta method. The scheme has been shown to achieve second order of accuracy, both in space and time. Another Lagrangian discontinuous Galerkin finite element method has been recently proposed in [49]. The method preserves discrete conservation in the presence of arbitrary mesh motion and thus obeys the Geometric Conservation Law (GCL).

In a series of articles [44–48] Maire et al. develop a general formalism to derive first and second order cell-centered Lagrangian schemes in multiple space dimensions and also on general polygonal grids. In [45] the time derivatives of the fluxes are obtained through the use of a node-centered solver which can be viewed as a multi-dimensional extension of the Generalized Riemann problem methodology introduced by Ben-Artzi and Falcovitz [3], Le Floch et al. [5, 33] and Titarev and Toro [62, 63, 66]. In their recent papers [10, 43] Cheng and Shu developed a class of cell centered Lagrangian finite volume schemes for solving the Euler equations which are based on high order essen-