

## Space-Time Discontinuous Galerkin Method for Maxwell's Equations

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**Abstract.** A fully discrete discontinuous Galerkin method is introduced for solving time-dependent Maxwell's equations. Distinguished from the Runge-Kutta discontinuous Galerkin method (RKDG) and the finite element time domain method (FETD), in our scheme, discontinuous Galerkin methods are used to discretize not only the spatial domain but also the temporal domain. The proposed numerical scheme is proved to be unconditionally stable, and a convergent rate  $\mathcal{O}((\Delta t)^{r+1} + h^{k+1/2})$  is established under the  $L^2$ -norm when polynomials of degree at most  $r$  and  $k$  are used for temporal and spatial approximation, respectively. Numerical results in both 2-D and 3-D are provided to validate the theoretical prediction. An ultra-convergence of order  $(\Delta t)^{2r+1}$  in time step is observed numerically for the numerical fluxes w.r.t. temporal variable at the grid points.

**AMS subject classifications:** 35Q61, 65M12, 65M60, 65N12

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## 1 Introduction

Finite element methods, including edge element methods and discontinuous Galerkin methods, have been widely used to solve time-harmonic Maxwell's equations [2,4,28] as well as time-dependent Maxwell's equations [8,9,11,15,20–27], due to their high order accuracy and flexibility in handling complicated domains. Traditionally, they were only used to discretize the spatial domain to produce a system of ordinary differential equations (in time  $t$ ), which was then solved by the finite difference or Runge-Kutta methods [8,11,15,25,27]. Towards this end, Makridakis and Monk proposed a fully discrete finite element method for Maxwell's equations and investigated the corresponding error estimates in [26]. Their approach resulted in a coupled non-symmetric and indefinite linear algebraic system involving both electric and magnetic fields. Later, Ciarlet Jr. and Zou [9] analyzed a fully discrete finite element approach for a second-order electric field equation derived from Maxwell's equations by eliminating the magnetic field. Both optimal energy-norm error estimate and optimal  $L^2$ -norm error estimate were obtained. When dispersive media were involved, Li proposed some fully discrete numerical schemes. Both mixed finite element method [20–22] and interior penalty discontinuous Galerkin method [23] are considered for spatial discretization. Since Maxwell's equations are a coupled system, a fully discrete scheme was proposed by Ma [25], aimed to reduce the computational cost by denoting the magnetic field explicitly in the numerical scheme.

The idea to discretize the temporal domain by finite element method is not something new in the literature. Actually it was proposed as early as in late 60's by Argyris and Scharpf [1], and Oden [30]. Since then the space-time finite element methods have been widely used to solve a variety of differential equations, e.g., see [3,5,16] for the implementation of time-continuous Galerkin finite element schemes. Some works on space-time finite element method for solving hyperbolic equations are available, see [29,32]. Recently, Tu et al., proposed a space-time discontinuous Galerkin cell vertex scheme to solve conservation law and time dependent diffusion equations [33]. This scheme is conceptually simpler than other existing DG-type methods. Nevertheless, to the best of our knowledge, the finite element method has not been used to discretize the temporal domain in fully discrete scheme for Maxwell's equations up to now.

On the other hand, time-discontinuous Galerkin methods were originally developed for the first order hyperbolic equations [19,31] and have been successfully applied to various hyperbolic and parabolic equations (see [12,16] and the references therein). They usually lead to some stable and higher-order accurate numerical schemes. Actually in [18,19], the time-discontinuous Galerkin method was first shown to be an A-stable, higher-order accurate ordinary differential equation solver. Furthermore, the time-discontinuous Galerkin framework seems conducive for the rigorous justification of the error estimates [18].

In [34] we introduced a semi-discrete locally divergence-free DG method for solving Maxwell's equations in dispersive media under a unified framework. After the discretization of the spatial domain, we obtained a Volterra integro-differential system in