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Numerical Methods for Balance Laws with Space Dependent Flux: Application to Radiotherapy Dose Calculation

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Abstract. The present work is concerned with the derivation of numerical methods to approximate the radiation dose in external beam radiotherapy. To address this issue, we consider a moment approximation of radiative transfer, closed by an entropy minimization principle. The model under consideration is governed by a system of hyperbolic equations in conservation form supplemented by source terms. The main difficulty coming from the numerical approximation of this system is an explicit space dependence in the flux function. Indeed, this dependence will be seen to be stiff and specific numerical strategies must be derived in order to obtain the needed accuracy. A first approach is developed considering the 1D case, where a judicious change of variables allows to eliminate the space dependence in the flux function. This is not possible in multi-D. We therefore reinterpret the 1D scheme as a scheme on two meshes, and generalize this to 2D by alternating transformations between separate meshes. We call this procedure projection method. Several numerical experiments, coming from medical physics, illustrate the potential applicability of the developed method.

AMS subject classifications: 35L65, 65N08, 92C99

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1 Introduction

This paper is concerned with the numerical approximation of the set of balance laws

$$\partial_\varepsilon \Psi^0(\mathbf{x},\varepsilon) - \nabla_x \cdot \Big(\frac{1}{\rho(\mathbf{x})} \mathbf{\Psi}^1(\mathbf{x},\varepsilon) \Big) = 0, \tag{1.1a}$$

$$\partial_\varepsilon \mathbf{\Psi}^1(\mathbf{x},\varepsilon) - \nabla_x \Big(\frac{1}{\rho(\mathbf{x})} D_e \Big(\frac{\mathbf{\Psi}^1(\mathbf{x},\varepsilon)}{\Psi^0(\mathbf{x},\varepsilon)} \Big) \Psi^0(\mathbf{x},\varepsilon) \Big) = T(\mathbf{x},\varepsilon) \mathbf{\Psi}^1(\mathbf{x},\varepsilon). \tag{1.1b}$$

As will be explained below, this system describes charged particle transport in tissue, and can be used as a novel method for dose calculation in radiotherapy. The particular challenge in the numerical approximation of this system of hyperbolic balance laws is that the flux function depends explicitly on a density $\rho(\mathbf{x})$. In addition, this density can vary over several orders of magnitude, from $\rho \sim 1$ (water) to $\rho \sim 10^{-3}$ (air). A standard discretization would therefore require a very small time step, which would make the computations infeasible. To overcome this problem, we develop a specific technique to deal with the strongly varying space-dependent flux. Several authors have worked on analysis and numerical methods for conservation laws for discontinuous flux. An overview can be found in the recent paper [45]. Our approach is new because it uses the specific structure of the discontinuous coefficient and introduces a method that is based on variable transformations.

Radiotherapy is the treatment of cancer and other diseases with a certain type of ionizing radiation. This radiation deposits energy that injures or destroys cells in the area being treated, by damaging their genetic material. Indeed, the incoming particles ionize atoms and make the cells unstable. Physicians want to determine the best beam setup to destroy the tumor while minimizing the damage to healthy tissue. To that end, one needs to predict and visualize the radiation dose in the patient's body before the treatment. The data used, which usually comes from a CT (Computer Tomography) scan, consist of 2D slices describing the density of the tissues.

Up to now many clinical dose calculation algorithms rely on semi-empirical models. They are based on explicit solutions to radiation problems in a very simplified geometry (e.g., the one-dimensional Fermi-Eyges theory [18]). These explicit solutions are then combined with experimental data to compute the central-axis dose (e.g., [26]). Although many improvements of Fermi-Eyges theory were performed, e.g., by including additional correction factors [1, 27, 28, 37], they still produce errors of up to 12% near inhomogeneities [29].

Currently, statistical Monte-Carlo simulation codes are entering clinical practice [14, 38, 39]. These perform direct simulations of individual particle tracks which result from a random sequence of free flights and interaction events. In this way random histories are generated. If their number is large enough macroscopic quantities can be obtained by averaging over the simulated histories [2]. Monte Carlo tools model physical processes very precisely and can handle arbitrary geometries without losing accuracy. Although