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Reduction of Linear Systems of ODEs with Optimal Replacement Variables

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To the memory of David Gottlieb

Abstract. In this exploratory study, we present a new method of approximating a large system of ODEs by one with fewer equations, while attempting to preserve the essential dynamics of a reduced set of variables of interest. The method has the following key elements: (i) put a (simple, ad-hoc) probability distribution on the phase space of the ODE; (ii) assert that a small set of *replacement variables* are to be unknown linear combinations of the not-of-interest variables, and let the variables of the reduced system consist of the variables-of-interest together with the replacement variables; (iii) find the linear combinations that minimize the difference between the dynamics of the original system and the reduced system. We describe this approach in detail for linear systems of ODEs. Numerical techniques and issues for carrying out the required minimization are presented. Examples of systems of linear ODEs and variable-coefficient linear PDEs are used to demonstrate the method. We show that the resulting approximate reduced system of ODEs gives good approximations to the original system. Finally, some directions for further work are outlined.

AMS subject classifications: 65M06, 65M30, 65M70

Key words: System reduction, optimal replacement variables, resolved variables, optimal prediction.

1 Introduction

The framework of the problem studied in this paper is a system of linear ordinary differential equations (ODEs)

$$z_t = Fz = \begin{pmatrix} F_0 & F_1 \\ F_2 & F_3 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad t>0, \tag{1.1}$$

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with appropriate initial conditions $z(t=0)$, and where $z=(x,y)\in\mathrm{R}^{m+n}$ is divided into a set of *resolved* variables $x\in\mathrm{R}^m$ that one wants to observe or calculate, and a set of *unresolved* variables $y\in\mathrm{R}^n$ that one doesn't need to observe, but which the dynamics of the resolved variables x depend on. Furthermore, it may be that $m\ll n$ or even n infinite, in which case it will not be computationally feasible to solve the full system of equations.

The goal of this study is to approximate (model) the dynamics of x in a computationally efficient way, with useful accuracy, without actually including the full set of unresolved quantities y in the modeling.

This is desirable in many situations: for example, many partial differential equations (PDEs), when discretized into a system of ODEs, require millions of degrees of freedom (DOF) to adequately approximate the dynamics. However, most of these DOF are usually of no interest. Examples of such PDEs include weather simulations and many simulations of fluid or aerodynamic flows. In the flow-around-an-aircraft example, an engineer would be mainly interested in bulk features such as the total lift and drag, or average vorticity as a function of time. A flow field detailed enough to actually resolve all of the dynamics would not be needed in many situations. Calculating solutions to these equations can require large amounts of computing resources, and a system reduction method such as the one studied here has the potential to reduce these resource required, or allow the fast solution of more complex problems.

One approach for system reduction has been developed by Chorin et al. [1, 2] and Gottlieb et al. [3], which has been called the *t-system* or *Optimal Prediction*. An overview of other approaches to the problem can be found in [4].

Much previous work, including much work by Chorin and associates, has revolved around approximate systems having only x as the state variables and eliminating the unresolved variables y. So then x has to somehow represent the dynamics of both the resolved and unresolved variables. Any information about the unresolved dynamics in the t-system must be the result of resolved-dynamics information being discarded or changed. This seems like a limited approach, that could not achieve high accuracy. We propose adding a set of "replacement variables" to the resolved variables to contain the unresolved-dynamics information. This set of variables would be smaller than the actual number of unresolved variables, since we only need to contain the part of the unresolved dynamics that effects the resolved variables. In Section 2, the framework of the ORV method is described in details and the expected error with ORV derived. In Section 3, the complicated form of the modeling error equation and its gradient which will be used later for finding the best replacement variables R, are studied and simplified. Several ways for normalizing the ORV error and orthogonality constraints of the ORV system are also discussed. The techniques of Lagrange multipliers and unconstrained minimization used for minimizing the expected error of the ORV system are presented in Section 4. In Section 5, numerical results for random linear systems of ODEs and a scalar variable-coefficient PDE (a heat equation), are presented to illustrate the potential and issues in the ORV method. Discussion and conclusions are given in Section 6. We outline some directions and questions for future research in Section 7.