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GALERKIN CHARACTERISTICS METHOD FOR CONVECTION-DIFFUSION PROBLEMS WITH MEMORY TERMS

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Abstract. We use the modified method of characteristics for solving nonlinear convection diffusion problems with memory terms. The convergence of approximation scheme is proved under minimal regularity assumptions on the velocity field and on the solution. The results are supported by numerical experiments for contaminant transport with diffusion and non-equilibrium sorption isotherms.

Key Words. nonlinear diffusion, convection-adsorption, method of characteristics.

1. Introduction

We consider the following mathematical model for convection diffusion with memory term

$$\begin{aligned} &\partial_t b(x,u) + \operatorname{div}(\bar{F}(t,x,u) - k\nabla u) = f(t,x,u,s), \\ &s(t,x) = \textstyle\int_0^t K(t,z)\psi(u(z,x))dz \end{aligned} \tag{1.1}$$

in $\Omega \times (0,T]$, $T < \infty$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\partial\Omega \in C^{1,1}$, see [26]. If Ω is convex, then $\partial\Omega$ is assumed to be Lipschitz continuous. We consider a Dirichlet boundary condition

$$u(t,x) = 0 \quad \text{on } I \times \partial\Omega, \quad I = (0,T], \tag{1.2}$$

together with the initial condition

$$u(0,x) = u_0(x) \quad x \in \Omega. \tag{1.3}$$

We assume $0 < \varepsilon \le \partial_s b(x,s) \le M < \infty$, $k > 0$ and suppose that f is sublinear in u, s and $\psi(z)$ is sublinear in z. The convection term $\bar{F}$ is Lipschitz continuous in u.

The mathematical model (1.1)-(1.3) is motivated by contaminant transport in porous media intensively studied in the last years, see [4, 9, 10, 11, 19, 20, 21, 1]

$$\begin{aligned} &\partial_t(\theta C + \rho S) + \operatorname{div}(\bar{q}C - D\nabla C) = 0, \\ &\rho\partial_t S = d(\psi(C) - S), \end{aligned} \tag{1.4}$$

where C is the concentration of the contaminant, $\bar{q}$ is the velocity field (Darcy), D is the diffusion matrix, ρ is the bulk density, ψ is the sorption isotherm of the porous media with porosity θ. Here, S is the mass of contaminant adsorbed by the unit mass of porous medium. The coefficient d describes the rate of adsorption. If $d \to \infty$, then an equilibrium sorption process occurs with $S = \psi(C)$ and hence, $b(s) = \theta s + \rho\psi(s)$ generates the parabolic term in (1.1) with $f \equiv 0$. If $d << \infty$,

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the sorption process becomes non-equilibrium. Then, we can eliminate S from the ODE and obtain

$$b(x,z) \equiv \theta(x)z,$$
$$f(t,x,u,s) = d\left(-\psi(u(t,x)) + s_0\,\mathrm{e}^{-\frac{d}{\rho}t} + d\,s\right), \quad K(t,z) = \mathrm{e}^{-\frac{d}{\rho}(t-z)}$$

in our model (1.1). The most common isotherms are $\psi(z) = \frac{c_1 z}{1+c_2 z}$ $(c_1, c_2 > 0)$ (Langmuir isotherm) or $\psi(z) = cz^p$ ($0 < p < 1$ and $c > 0$) (Freundlich isotherm). In the case of the Freundlich isotherm, in the equilibrium mode we obtain the model

$$b(x,z) \equiv \theta(x)z + \rho z^p, \;\; f(t,x,u,s) \equiv 0,$$

which violates $\partial_z b(x,z) < M < \infty$. In such a case, our model can be considered as an approximation of the more general case including $\partial_z b(x,z) = \infty$ in some points z, see [17]. However, such a problem does not occur in the non-equilibrium model even if ψ is of Freundlich isotherm type (not Lipschitz continuous). Our model (1.1) includes locally both equilibrium (in the Freundlich isotherm type we have the approximation of the parabolic term) and non-equilibrium adsorption. Moreover, it is a convection dominated diffusion model. For simplicity, from now on we will drop the variables x in the terms b, $\bar{F}$, f.

The outline of this paper is as follows. In section 2, we define our numerical scheme. In section 3, we prove its convergence and adress related issues. Section 4 deals with the error estimate for our scheme. In section 5, we discuss the numerical implementation and present a variety of 1D and 2D examples.

2. Definition of the scheme

Our approximation scheme is as follows: Let $u_i \approx u(t_i, x)$, $t_i = i\tau$, $\tau = \frac{T}{n}, (n \in \mathbb{N})$. At time level $t = t_i$, we determine u_i successively for $i = 1, \ldots, n$ from the linear elliptic problem of the form

$$\begin{aligned} b'(u_{i-1})\left(\tfrac{u_i - u_{i-1}\circ\varphi^i}{\tau}\right) - k\Delta u_i &= f(t_i, u_{i-1}, s_i) - \mathrm{div}_x \bar{F}(t_i, u_{i-1}) \\ &\equiv H(t_i, u_{i-1}, s_i) \end{aligned} \tag{2.5}$$

$$u_i = 0 \quad \text{on } \partial\Omega, \qquad s_i = \sum_{j=1}^{i-1} \alpha_{ij}\psi(u_j)\tau$$

where

$$\varphi^i = x - \tau\omega_h * \left(\tfrac{\bar{F}'_u(t_i, u_{i-1})}{b'(u_{i-1})}\right), \quad \alpha_{ij} = \tfrac{1}{\tau}\int_{t_{j-1}}^{t_j} K(t_i, z)dz \tag{2.6}$$

and $\omega_h * g$ is the convolution of the mollifier ω_h with $g \in L_\infty(\Omega)$. As a mollifier we can take $\omega_h(x) = \omega_1(\frac{x}{h})\frac{1}{h^N}$ where

$$\omega_1(x) := \begin{cases} \frac{1}{\kappa}\exp(\frac{|x|^2}{|x|^2-1}) & \text{for } |x| \le 1 \\ 0 & \textit{otherwise} \end{cases}, \qquad \kappa = \int_{x \le 1} \exp(\frac{|x|^2}{|x|^2 - 1})\,.$$

The approximation scheme (2.5) represents the approximation of the two processes in the time interval $t \in (t_{i-1}, t_i)$ where the composition $u_{i-1}\circ\varphi^i$ represents the transport part of the concentration profile u_{i-1} along the approximated characteristics φ^i and the diffusion process is approximated by the implicit Euler scheme. We note that for the transport equation

$$\partial_t u + \bar{q}\cdot\nabla u = 0,$$