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REVIEW ARTICLE

Fast Numerical Methods for Stochastic Computations: A Review

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Abstract. This paper presents a review of the current state-of-the-art of numerical methods for stochastic computations. The focus is on efficient high-order methods suitable for practical applications, with a particular emphasis on those based on generalized polynomial chaos (gPC) methodology. The framework of gPC is reviewed, along with its Galerkin and collocation approaches for solving stochastic equations. Properties of these methods are summarized by using results from literature. This paper also attempts to present the gPC based methods in a unified framework based on an extension of the classical spectral methods into multi-dimensional random spaces.

AMS subject classifications: 41A10, 60H35, 65C30, 65C50

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1 Introduction

The purpose of this paper is to present an overview of the recent development of numerical methods for stochastic computations, with a focus on fast algorithms suitable for large-scale complex problems. This field has received an increasing amount of attention recently and is developing at a fast pace with new results emerging as the paper is under writing. Therefore this paper is not an attempt to present an exhaustive review of all available results, which is a goal almost impossible to achieve. The focus is rather on the popular methods based generalized polynomial chaos (gPC) methodology. We will present the framework and properties of the methods by using (almost) exclusively published work and demonstrate that the methods can be considered as a natural extension of deterministic spectral methods into random spaces.

1.1 Uncertainty quantification

The ultimate objective of numerical simulations is to predict physical events or the behaviors of engineered systems. Extensive efforts have been devoted to the development of accurate numerical algorithms so that simulation predictions are reliable in the sense that numerical errors are well under control and understood. This has been the primary goal of numerical analysis, which remains an active research branch. What has been considered much less in the classical numerical analysis is the understanding of impacts of errors, or uncertainty, in "data" such as parameter values, initial and boundary conditions.

The goal of uncertainty quantification (UQ) is to investigate the impact of such errors in data and subsequently to provide more reliable predictions for practical problems. This topic has received an increasing amount of attention in the past years, especially in the context of complex systems where mathematical models can serve only as simplified and reduced representations of the true physics. Although many models have been successful in revealing quantitative connections between predictions and observations, their usage is constrained by our ability of assigning accurate numerical values to various parameters in the governing equations. Uncertainty represents such variability in data and is ubiquitous because of our incomplete knowledge of the underlying physics and/or inevitable measurement errors. Hence in order to fully understand simulation results and subsequently the true physics, it is imperative to incorporate uncertainty from the beginning of the simulations and not as an afterthought.

1.1.1 Burgers' equation: An illustrative example

Instead of engaging in an extensive discussion on the significance of UQ, which there are many, let us demonstrate the impact of uncertainty via a simple example of the viscous Burgers' equation,

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & x \in [-1,1], \\ u(-1) = 1, \quad u(1) = -1, \end{cases} \tag{1.1}$$