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Modes of a Plasma-Filled Waveguide Determined by a Numerical *hp* Method

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Abstract. We present the application of the recent physics-conforming COOL method [2, 4] to the eigenvalue problem of a cylindrical waveguide filled with unmagnetized plasma. Using the Fourier transform only *along* the waveguide and not in poloidal direction, this is a relevant test case for a numerical discretization method in two dimensions (radial and poloidal). Analytically, the frequency spectrum consists of discrete electromagnetic parts and, depending on the electron density profile of the plasma, of infinitely degenerate and/or continuous, essentially electrostatic parts. If the plasma is absent, the latter reduces to the infinitely degenerate zero eigenvalue of electrostatics. A good discretization method for the Maxwell equations must reproduce these properties. It is shown here that the COOL method meets this demand properly and to very high precision.

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Key words: COOL method, physics-conforming, continuous spectrum, plasma waveguide, spectral method.

1 Introduction

The paper describes the generalization of a finite element approach that has successfully been used in the past to compute the stability behaviour of Tokamaks [9, 10] and Alfvén wave heating [3] of such fusion devices. In these applications the fundamental

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numerical problems were the elimination of the so-called spectral pollution arising from standard finite elements and the poor precision of pollution-free conforming finite elements [9]. This mathematically non-conforming numerical approach proposed in [8, 9] not only eliminated spectral pollution but also delivered high precision solutions with superconvergence properties although only polynomials of order $p=1$ and 0 had been used.

Recently, the COOL (Constraints Oriented Library) method [2] has generalized this older approach to higher order polynomial degrees $p>1$. In this hp method (h being a measure of the discretization) each term of the variational form is represented by a polynomial of degree $p-1$ in each direction, and is discontinuous across all element borders. Internal constraints such as $\vec{\nabla}\cdot\vec{B}=0$ or $\vec{\nabla}\times\vec{E}=0$ can then be identically satisfied. This enables reaching lower energy levels, thus, approximating better the underlying physics.

Mathematically, the magnetohydrodynamic (MHD) Alfvén waves have a lot in common with the electrostatic Langmuir oscillations in unmagnetized cold plasma [3]. In a waveguide filled with cold plasma the electromagnetic waveguide modes are, in general, coupled among themselves and to the Langmuir oscillations. The degree of coupling depends on the mode frequency and the density gradient of the plasma. It is therefore impossible to impose explicitly conditions like $\vec{\nabla}\times\vec{E}=0$ for dominantly electrostatic modes or $\vec{\nabla}\cdot\vec{E}=0$ for electromagnetic ones, respectively; some modes may show such a character to a high degree and others not. Their property must be the result of the calculation and cannot be imposed externally. In the limiting case of vanishing plasma density, the frequency of the electrostatic oscillation tends to zero: These eigensolutions correspond to the electrostatic solutions of Maxwell's equations in vacuo. If standard finite elements are used in this case, the infinitely degenerate zero eigenvalues appear as a series of discrete eigenvalues with an accumulation point at zero, an unacceptable result! With standard triangular elements, the number of constraints to be satisfied doubles. There are not enough variables to satisfy all of them, and spectral pollution appears. Edge elements [5], another numerical approach based on triangles, are known to give results without spectrum pollution. However, it is expected that the precision needed in MHD cannot be achieved.

After presenting Maxwell's equations for a bounded, unmagnetized plasma in a cylindrical waveguide of radius $r=1$, the COOL method is briefly presented. Specifically, a new type of basis functions is presented. They are zero in most of the Gauss points used for the numerical integration, thus reducing the number of operations to compute the matrix elements. To show the efficiency of the COOL method, a cylindrical infinitely long plasma configuration is considered. With a Fourier analysis in the angular and longitudinal directions, the homogeneous plasma wave spectrum is analytically tractable. Numerically, this 1D geometry is considered as a 2D geometry, the angular variation is numerically resolved. In fact, numerical difficulties appearing in a toroidal geometry relevant to ITER (www.iter.org) computations also appear in a cylinder: The periodicity in the longitudinal direction simulates the toroidal periodicity, and the angular direction behaves like the poloidal angle in a torus. A variable transformation done to satisfy the