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Numerical Issues in the Implementation of High Order Polynomial Multi-Domain Penalty Spectral Galerkin Methods for Hyperbolic Conservation Laws

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Abstract. In this paper, we consider high order multi-domain penalty spectral Galerkin methods for the approximation of hyperbolic conservation laws. This formulation has a penalty parameter which can vary in space and time, allowing for flexibility in the penalty formulation. This flexibility is particularly advantageous for problems with an inhomogeneous mesh. We show that the discontinuous Galerkin method is equivalent to the multi-domain spectral penalty Galerkin method with a particular value of the penalty parameter. The penalty parameter has an effect on both the accuracy and stability of the method. We examine the numerical issues which arise in the implementation of high order multi-domain penalty spectral Galerkin methods. The coefficient truncation method is proposed to prevent the rapid error growth due to round-off errors when high order polynomials are used. Finally, we show that an inconsistent evaluation of the integrals in the penalty method may lead to growth of errors. Numerical examples for linear and nonlinear problems are presented.

AMS subject classifications: 65M70, 65M12, 65M60, 65L07

Key words: High order polynomial Galerkin methods, penalty boundary conditions, discontinuous Galerkin methods, hyperbolic conservation laws, round-off errors, truncation methods.

1 Introduction

Spectral and discontinuous Galerkin methods are widely used for the numerical solution of hyperbolic conservation laws [1, 2, 8, 12, 13, 17, 19]. These methods seek a polynomial approximation of the solution for which the projected residual of the differential equation

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to the polynomial space vanishes. Traditionally, spectral Galerkin methods (sGM) have used high order polynomials on one element, while discontinuous Galerkin methods (dGM) use lower order polynomials on many elements. However, multi-domain sGM exist and are known as spectral element methods. In order to increase the accuracy of the approximation, these methods can use more smaller elements (h-refinement) or raise the degree of the polynomial in each element (p refinement). High order polynomials have numerical issues such as sensitivity to roundoff errors, so it is important to carefully study their effects on accuracy and stability when used in multi-domain penalty spectral Galerkin methods.

The penalty formulation penalizes the boundary or interface conditions at each element by introducing some term which depends on a parameter. This penalty parameter allows a great deal of flexibility, as it can change over space and time. We demonstrate the advantages of the flexibility in the choice of penalty parameter, especially in the case where an inhomogeneous grid system is used. An inhomogeneous grid can be due to a difference in element size or in polynomial order at each element, but this type of grid is subject to non-physical reflecting or dispersive modes which may appear in the solution. By modifying the penalty conditions near the grid discontinuity, we show that the sGM can reduce the non-physical modes while computing the other elements efficiently and accurately. We further consider the effects of the penalty method on the stability and accuracy. In this context, we show that the dG formulation is a special case of the penalty multi-domain sGM, for a particular value of the penalty parameter.

Next we discuss the effect of round-off errors for high order sGM. These round-off error effects can arise from the ill-conditioned mass matrix for high order polynomials, and the numerically inconsistent evaluations of the mass matrix and the load vector. The coefficient truncation method is introduced to reduce round-off errors. This method truncates high order coefficients in the solution of the linear system which do not show rapid decay, thus preventing error growth. To prevent roundoff error without losing high order information, we truncate not the coefficients of the polynomial but rather the right-hand side of the system, after Gaussian elimination is performed but before back-substitution. The second round-off error effect we explore is the error resulting from inconsistent evaluations of the two sides of the equation. While the right-hand-side of the linear system is evaluated by quadrature, the left-hand side can usually be computed exactly. In the case of orthogonal polynomials, this matrix is a diagonal matrix which simplifies the process of solving the system. However, computing one side of the equation exactly and the other by quadrature results in inconsistency errors which rise when the polynomial order is raised. We show these numerical inconsistency errors, and their dependence on the penalty parameter, in numerical computations.

The paper is structured as follows. In Section 2, we formulate the multi-domain penalty sGM and demonstrate the effect of the penalty terms on the stability and accuracy. The penalty sGM with inhomogeneous grid and the flexibility of the penalty methods are discussed. The equivalence of the dGM to the sGM is shown as well. In Section 3, the effect of round-off errors on the high order sGM is discussed. The coefficient