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Error Control in Multi-Element Generalized Polynomial Chaos Method for Elliptic Problems with Random Coefficients

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Abstract. We develop the theory for a robust and efficient adaptive multi-element generalized polynomial chaos (ME-gPC) method for elliptic equations with random coefficients for a moderate number ($\mathcal{O}(10)$) of random dimensions. We employ low-order ($p \leq 3$) polynomial chaos and refine the solution using adaptivity in the parametric space. We first study the approximation error of ME-gPC and prove its hp-convergence. We subsequently generate local and global *a posteriori* error estimators. In order to resolve the error equations efficiently, we construct a reduced space using much smaller number of terms in the enhanced polynomial chaos space to capture the errors of ME-gPC approximation. Based on the *a posteriori* estimators, we propose and implement an adaptive ME-gPC algorithm for elliptic problems with random coefficients. Numerical results for convergence and efficiency are also presented.

AMS subject classifications: 65C20, 65C30

Key words: Stochastic PDE, a posteriori error estimate, elliptic problems, adaptive numerical methods, uncertainty quantification.

1 Introduction

Error control in large-scale simulations is based primarily on a combination of heuristic algorithms and physical considerations, often ignoring the mathematical properties of the governing equations. Progress has been made, however, especially for finite element discretizations, where techniques such as adaptive mesh refinement based on *a posteriori* error estimation (see [1, 2] and references therein), and adaptive modeling refinement [3, 4] have been developed and applied to physical applications to reduce the simulation

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errors. These techniques are used for deterministic numerical simulations mostly in two- and less often in three-dimensions.

Our interest in this work is to develop error control methods to high-dimensional stochastic problems, where the input data, e.g., transport coefficients, boundary conditions or forcing and interaction terms, are modeled as random processes. A general framework in modeling such stochastic problems has been developed in [5–9] where Galerkin type expansions were employed in conjunction with a random trial basis to obtain a deterministic set of equations that can subsequently be solved with standard numerical methods. In particular, in [8] we developed an adaptive multi-element generalized polynomial chaos (ME-gPC) method.

In this paper we consider elliptic partial differential equations with stochastic coefficients which have many physical applications, e.g., random vibrations, composite materials, etc.; see [5, 10–12] and the references therein. Our objective is to set the theoretical foundations for ME-gPC and derive rigorous algorithms for error control in solving such equations.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of Ω, and P is a probability measure. Let D be a bounded, connected, open subset of $\mathbb{R}^d (d=1,2,3)$ with a Lipchitz continuous boundary ∂D. We consider the following stochastic linear boundary value problem: find a stochastic function, $u:\Omega\times\overline{D}\to\mathbb{R}$, such that almost surely (a.s.) the following equation holds:

$$
\begin{aligned}
&-\nabla\cdot(a(\boldsymbol{x};\omega)\nabla u(\boldsymbol{x};\omega))=f(\boldsymbol{x}) \quad \text{in } D,\\
&u(\boldsymbol{x};\omega)=0 \quad \text{on } \partial D,
\end{aligned}
\tag{1.1}
$$

where $a(\boldsymbol{x};\omega)$ is a second-order random process satisfying the following assumption:

Assumption 1.1. Let $a(\boldsymbol{x};\omega)\in L_\infty(D;\Omega)$ be strictly positive with lower and upper bounds a_{min} and a_{max}, respectively,

$$
0<a_{min}<a_{max} \text{ and } \Pr(a(\boldsymbol{x};\omega)\in[a_{min},a_{max}],\forall\boldsymbol{x}\in\overline{D})=1. \tag{1.2}
$$

To obtain reliable simulation results for this problem in physical applications, we need to quantify the uncertainty associated with the random inputs and control the approximation errors. The traditional approach to deal with uncertainty is the Monte Carlo method and its variants, which rely on a relative large amount of realizations of the random solution field. These methods are not sensitive to the number of random dimensions but suffer from a relative low convergence rate. To this end, some non-sampling methods, such as perturbation methods [11] and second-moment analysis [13, 14] have been developed. These methods are usually restricted to systems with relatively small number of random inputs and outputs. Recently, another alternative of the non-sampling methods, the polynomial chaos method, has received considerable attention. Polynomial chaos is based on a set of basis functionals and a Galerkin projection, which is also called stochastic Galerkin method in the literature. The polynomial chaos bases can be classified based