INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 4, Pages 659–679

NUMERICAL SOLUTIONS OF STOCHASTIC DIFFERENTIAL DELAY EQUATIONS WITH JUMPS

GUIHUA ZHAO, MINGHUI SONG AND MINGZHU LIU

(Communicated by Ed Allen)

Abstract. In this paper, the semi-implicit Euler (SIE) method for the stochastic differential delay equations with Poisson jump and Markov switching (SDDEwPJMSs) is developed. We show that under global Lipschitz assumptions the numerical method is convergent and SDDEwPJMSs is exponentially stable in mean-square if and only if for some sufficiently small step-size Δ the SIE method is exponentially stable in mean-square. We then replace the global Lipschitz conditions with local Lipschitz conditions and the assumptions that the exact and numerical solution have a bounded pth moment for some $p > 2$ and give the convergence result.

Key Words. Poisson jump, Lipschitz condition, semi-implicit Euler method, exponential stability, convergence.

1. Introduction

Stochastic modeling has come to play an important role in many branches of science and industry and there are significant literatures that have been done concerning approximate schemes for stochastic differential equations (SDEs) with Markov switching [8, 12] or SDEs with Poisson jump [5, 6, 7].

In general, the future state of a system depends on the present and past states. Hence, it is more significant to consider stochastic differential delay equations with Poisson jump and Markov switching (SDDEwPJMSs). As many other equations, SDDEwPJMSs cannot be solved analytically. Thus, it is necessary to develop numerical methods and to study the properties of these methods. Finite time convergence analysis of an Euler scheme is given in [13]. In this work, we consider the finite time convergence of SIE method, the exponential mean-square stability of analytic and SIE numerical solutions.

Throughout this paper, we let $W(t)$ be a d-dimensional Brownian motion, $N(t)$ be a scalar Poisson process with intensity λ and independent of the Brownian motion. Also we let $r(t)$, $t \geq 0$ be a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$. The corresponding generator is denoted $\Gamma = (\gamma_{ij})_{N\times N}$, so that

$$\mathbb{P}\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) : & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) : & \text{if } i = j, \end{cases}$$

Received by the editors August 7, 2008 and, in revised form, June 24, 2009.
2000 *Mathematics Subject Classification.* 65C30, 65L20, 60H10.
This research is supported by the NSF of P.R. China (No.10671047).

where $\delta > 0$. Here γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij} \geq 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. Assume the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$ and Poisson jump $N(\cdot)$. We note that almost every sample path of $r(\cdot)$ is right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

In this paper, we need to work on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions. To construct such a filtration, we denote by $\mathcal{N}$ the collection of $\mathbb{P}$-null sets, that is $\mathcal{N} = \{A \in \mathscr{F} : \mathbb{P}(A) = 0\}$, For each $t \geq 0$, define $\mathscr{F}_t = \sigma(\mathcal{N} \cup \sigma(B(s), r(s), N(s) : 0 \leq s \leq t))$.

We will use $|\cdot|$ to denote the Euclidean norm of a vector and the trace norm of a matrix and $< \cdot, \cdot >$ to denote the scalar product. We will denote the indicator function of a set G by I_G and denote by $L^2_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ the family of $\mathscr{F}_t$-measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ such that

$$||\varphi||^2_{\mathbb{E}} := \sup_{-\tau \leq u \leq 0} \mathbb{E}|\varphi(u)|^2 < \infty.$$

For $\mu \in \mathbb{R}$, $\mathrm{In}[\mu]$ denote the integer part of μ. In this paper we consider the following n-dimensional SDDEwPJMSs

$$\begin{cases} dx(t) = f(t, x(t), x(\tau(t)), r(t))dt + g(t, x(t), x(\tau(t)), r(t))dW(t) \\ \qquad\qquad + h(t, x(t), x(\tau(t)), r(t))dN(t), \qquad\qquad t \geq 0, \\ x(t) = \varphi(t), \ r(0) = r_0, \qquad\qquad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau(t)$ satisfy:
there exists a positive constant ρ such that

$$-\tau \leq \tau(t) < t, \text{ and } |\tau(t) - \tau(s)| \leq \rho|t - s|, \ \forall t, s \geq 0, \tag{1.2}$$

and $\varphi(t) \in L^2_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$ which is uniformly Hölder continuous with exponent $\gamma \in (0, 1]$, that is, there exists a constant $M > 0$ such that for all $-\tau \leq s < t \leq 0$

$$\mathbb{E}|\varphi(t) - \varphi(s)|^2 \leq M(t - s)^\gamma. \tag{1.3}$$

We also assume that

$$a(t, 0, 0, i) = 0 \ \forall i \in \mathbb{S}, \ a = f, g, h, \tag{1.4}$$

so Eq. (1.1) admits the zero solution $x(t) = 0$.

To define the SIE approximate solution, we will need the following lemma (see [1]).

Lemma 1.1. *Given $\Delta > 0$, let $r^\Delta_k = r(k\Delta)$ for $k \geq 0$. Then $\{r^\Delta_k, k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$\mathbb{P}(\Delta) = (\mathbb{P}_{ij}(\Delta))_{N \times N} = e^{\Delta \Gamma}. \tag{1.5}$$

Given a fixed step size $\Delta > 0$ and the one-step transition probability matrix $\mathbb{P}(\Delta)$ in (1.5), the discrete Markov chain $\{r^\Delta_k, k = 0, 1, 2, \ldots\}$ can be simulated as follows [8]. Let $r^\Delta_0 = r_0$ and compute a pseudo-random number ξ_1 from the uniform