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Partial Pricing Rule Simplex Method with Deficient Basis[†]

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Abstract. A new partial pricing column rule is proposed to the basis-deficiency-allowing simplex method developed by Pan. Computational results obtained with a set of small problems and a set of standard NETLIB problems show its promise of success.

Key words: Linear programming; simplex method; deficient basis; partial pricing.

AMS subject classifications: 90C05

1 Introduction

We are concerned with the linear programming problem in the standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned} \tag{1}$$

where $A \in R^{m\times n}$ with $m < n$, $b \in R^m, c \in R^n, 1 \leq \text{rank}(A) \leq m$. It is assumed that the cost vector c, the right-hand b, and $A's$ columns and rows are nonzero, and that the equation $Ax = b$ is consistent.

It is widely accepted that pivot rules used in the simplex method affect the number of iterations required for solving linear programming problems. Much effort has therefore been made in the past on finding good pivot rules to improve the efficiency of the underlying method [1-10]. It is notable that although it usually involves more iterations, a partial pricing rule for column selection, like that used in the MINOS, involves less computational work per iteration, and consequently leads to less overall running time than the conventional full pricing.

The basis-deficiency-allowing simplex method for linear programming was developed by Pan [1] in 1997. A key feature of this method is that it uses a generalized basis allowing deficiency. Computational results obtained with this method, in which the full pricing rule is used, are

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indeed very encouraging. Therefore, the following questions naturally arise: Can we introduce some partial pricing to improve the new algorithm? If so, how to do it and what is its efficiency? In this paper, we propose a new partial pricing rule for column selection for the basis-deficiency-allowing simplex method to improve its efficiency further. We report computational results obtained with a set of small problems and a set of standard NETLIB problems, and show the rule's promise of success.

2 Preliminaries

For this presentation being self-contained, we first present the basis-deficiency-allowing simplex method briefly [1]. A basis is a submatrix consisting of any $m1$ linearly independent set of A's columns, whose range space includes b. If $m1 = m$, it is a normal basis; otherwise, it is a deficient basis.

Let B be a basis with $m1$ columns and let N be nonbasis, consisting of the remaining $n - m1$ columns. Define the ordered basic and nonbasic index sets respectively by

$$J_B = \{j_1, \ldots, j_{m1}\} \quad \text{and} \quad J_N = \{k_1, \ldots, k_{n-m1}\}$$

Thus we have

$$\begin{aligned} A &= [B, N] = [a_{j_1}, \ldots, a_{j_{m1}}; a_{k_1} \ldots, a_{k_{n-m1}}] \\ c^T &= [c_B^T, c_N^T] = [c_{j_1}, \ldots, c_{j_{m1}}; c_{k_1} \ldots, c_{k_{n-m1}}] \\ x^T &= [x_B^T, x_N^T] = [x_{j_1}, \ldots, x_{j_{m1}}; x_{k_1} \ldots, x_{k_{n-m1}}] \end{aligned}$$

Then program (1) can be written as

$$\begin{aligned} \min \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad & B x_B + N x_N = b \\ & x_B \geq 0, \ x_N \geq 0. \end{aligned} \tag{2}$$

Given the QR decomposition $Q^T B$=R, where $Q \in R^{m\times m}$ is orthogonal and $R \in R^{m \times m1}$ is upper triangular. Let Q and R be partitioned as $Q = [Q_1, Q_2]$ and $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$, where $Q_1 \in R^{m\times m_1}, Q_2 \in R^{m\times(m-m1)}, R_1 \in R^{m_1\times m_1}$ is upper triangular, and $0 \in R^{(m-m1)\times m_1}$ is the zero matrix. The canonical matrix may be partitioned as

$$[Q^T B \quad Q^T N \quad Q^T b] = \begin{bmatrix} R_1 & Q_1^T N & Q_1^T b \\ 0 & Q_2^T N & 0 \end{bmatrix}.$$

The associated basic solution is then

$$\bar{x}_N = 0, \quad \bar{x}_B = R_1^{-1} Q_1^T b, \tag{3}$$

the corresponding objective value is $f = c_B^T R_1^{-1} Q_1^T b$, and the corresponding reduced cost is

$$\bar{z}_N = c_N - N^T Q_1 R_1^{-T} c_B. \tag{4}$$

Let us take a single iteration. Assume that the current basic solution, say (3), is feasible, i.e., $\bar{x}_B \geq 0$. If in addition, it holds that $\bar{z}_N \geq 0$, then we are done. Suppose this is not the case. Conventionally, a nonbasic column is selected to enter the basis by Dantzig's original rule, i.e.,

$$q = \text{Argmin}\{\bar{z}_{k_j} \mid j = 1, \ldots, n - m1\} \tag{5}$$