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Three-Step Iterative Sequences with Errors for Uniformly Quasi-Lipschitzian Mappings†

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Abstract. The purpose of this paper is to investigate some sufficient and necessary conditions for three-step Ishikawa iterative sequences with error terms for uniformly quasi-Lipschitzian mappings to converge to fixed points. Our results extend and improve the recent ones announced by Liu [3, 4], Xu and Noor [5], and many others.

Key words: Uniformly quasi-Lipschitzian mappings; three-step Ishikawa iterative; fixed point.

AMS subject classifications: 47H05, 47H09, 49M05

1 Introduction

Throughout this paper, we assume that E is a real Banach space and C is a nonempty convex subset of E. Let $F(T)$ and N denote the set of fixed points and the natural number set, respectively. We recall the following definitions:

Definition 1.1. Let $T : C \to C$ be a mapping:

1. T is said to be *uniformly quasi-Lipschitzian* if there exists $L \in [1, +\infty)$, such that $\|T^n x - p\| \leq L\|x - p\|$, for all $x \in C, p \in F(T)$ and all $n \in N$.

2. T is said to be *uniformly L-Lipschitzian* if there exists $L \in [1, +\infty)$, such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$, and $n \in N$.

3. T is said to be *asymptotically quasi-nonexpansive* if there exists $k_n \in [1, +\infty)$ with $\lim_{n\to+\infty} k_n = 1$, such that $\|T^n x - p\| \leq k_n\|x - p\|$, for all $x \in C, p \in F(T)$ and all $n \in N$.

From the above definitions, it follows that if $F(T)$ is nonempty, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian, and an asymptotically quasi-nonexpensive mapping must be uniformly quasi-Lipschitzian. But the converse does not hold.

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Definition 1.2. Let E be a normed linear space, C be a nonempty convex subset of E, and $T : C \to C$ a given mapping. Then for arbitrary $x_1 \in C$, the iterative sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined by

$$\begin{cases} z_n = (1-\gamma_n-\nu_n)x_n + \gamma_n T^n x_n + \nu_n u_n, & n \geq 1 \\ y_n = (1-\beta_n-\mu_n)x_n + \beta_n T^n z_n + \mu_n v_n, & n \geq 1 \\ x_{n+1=}(1-\alpha_n-\lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n, & n \geq 1, \end{cases} \qquad \text{(TSISE)}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}, \{\nu_n\}$ are appropriate sequences in $[0, 1]$, is called the three-step Ishikawa iterative sequence with error terms of T.

We note that the usual modified Ishikawa and Mann iterations are special cases of the above three-step iterative scheme. If $\gamma_n = \nu_n \equiv 0$, then (TSISE) reduces to the usual modified Ishikawa iterative scheme with errors,

$$\begin{cases} y_n = (1-\beta_n-\mu_n)x_n + \beta_n T^n x_n + \mu_n v_n, & n \geq 1 \\ x_{n+1=}(1-\alpha_n-\lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n, & n \geq 1, \end{cases} \qquad \text{(MSISE)}$$

where $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}, \{\nu_n\}$ are appropriate sequences in $[0, 1]$.

For $\beta_n = \mu_n \equiv 0$ and $\gamma_n = \nu_n \equiv 0$, (TSISE) reduces to the usual modified Mann iterative scheme with errors,

$$\begin{cases} x_1 \in C \\ x_{n+1=}(1-\alpha_n-\lambda_n)x_n + \alpha_n T^n x_n + \lambda_n w_n, & n \geq 1, \end{cases} \qquad \text{(MMISE)}$$

where $\{w_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\lambda_n\}$ are appropriate sequences in [0,1].

In 1973, Petryshyn and Williamson in [1] proved a sufficient and necessary condition for Picard iterative sequences and Mann iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the result of [1] and gave a sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 2001, Liu [3, 4] extended the above results and obtained some sufficient and necessary conditions for Ishikawa iterative sequences with errors members for asymptotically quasi-nonexpansive mappings to converge to fixed points. Recently Xu and Noor [5] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings.

The purpose of this paper is to investigate some sufficient and necessary conditions for three-step Ishikawa iterative sequences with error terms for uniformly quasi-Lipschitzian mappings to converge to fixed points. Our results presented in this paper extend and improve the recent ones announced by Liu [3, 4], Xu and Noor [5], and many others to uniformly quasi-Lipschitzian mappings.

In the sequel, we shall need the following lemma:

Lemma 1.1 ([6]; Lemma 1)**.** *Let $\{a_n\}, \{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\Sigma_{n=1}^{\infty}\delta_n < \infty$ and $\Sigma_{n=1}^{\infty}b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ is a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.