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Superconvergence of a Nonconforming Finite Element Approximation to Viscoelasticity Type Equations on Anisotropic Meshes†

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Abstract. The main aim of this paper is to study the approximation to viscoelasticity type equations with a Crouzeix-Raviart type nonconforming finite element on the anisotropic meshes. The superclose property of the exact solution and the optimal error estimate of its derivative with respect to time are derived by using some novel techniques. Moreover, employing a postprocessing technique, the global superconvergence property for the discretization error of the postprocessed discrete solution to the solution itself is studied.

Key words: Viscoelasticity type equations; Crouzeix-Raviart type element; superclose and superconvergence properties; postprocessing; anisotropic meshes.

AMS subject classifications: 65N25, 65N15, 65N30

1 Introduction

It is well-known that the subdivision T_h to domain Ω is demanded to satisfy the regular condition or quasi-uniform assumption in conventional finite element method [1], i.e., $h_K/\rho_K \leq c$ or $h/\overline{h} \leq c$, where h_K and ρ_K are the diameter of $K \in T_h$ and the supremum of the diameters of all circles inscribed in K, respectively, $h = \max_K h_K$, $\overline{h} = \min_K \overline{h}_K$, and c is a positive number independent of K and h. However, the domain considered may be narrow and irregular. For example, in modeling a gap between rotor and stator in an electrical machine, or in modeling a cartilage between a joint and hip, if we employ the regular partition of the domain then the cost of calculation will be very high. It is an obvious idea to employ the anisotropic triangulation which has fewer degrees of freedoms than the traditional triangulation.

Anisotropic element K is characterized by $h_K/\rho_K \longrightarrow \infty$ where the limit is considered as $h \longrightarrow 0$. Consequently, some basic theories and techniques of the classical finite element methods cannot be applied in such cases. For example, when the consistency error of nonconforming element is estimated with traditional technique, $\frac{meas(F)}{measK}$ is presented and may be infinite if F

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is of long edge. In such cases new techniques will be employed to obtain convergence. On the other hand, the Sobolev interpolation theory can not be used directly. Hence, the well-posedness, stability and LBB condition (one of the key points with mixed finite element methods) of interpolation operator are very difficult. Apel *et al.* [2, 3] presented an anisotropic interpolation theorem that can be used to check the anisotropy of an element. But it is not convenient in practical computations. Chen *et al.* [4] supplied an improved one which is much easier than that of [2, 3], which was used to check the anisotropy of Lagrange type, Hermite type, Crouzeix-Raviart type, quasi-Wilson element, ACM's element, Carey element, and so on (refers to [4, 6-9]). The above models and results show that it is not necessary to require meshes to satisfy the regularity conditions needed in classical finite element methods. Therefore, the study about anisotropic finite elements becomes an important issue in theoretical analysis and engineering practices and there have been many useful results [2-9] on such aspect in recent years.

The global superconvergence of the Lagrange type finite elements for viscoelasticity type equations was studied in [10] on regular meshes.

In this paper, the approximation of a Crouzeix-Raviart type nonconforming finite element [8] for the viscoelasticity equations is studied on anisotropic meshes. The superclose property of the exact solution and the optimal error estimate of its derivative with respect to time are obtained, which are the same as those in [10]. Furthermore, the global superconvergence property is also obtained by employing a postprocessing technique. Thus the results of this paper can be regarded as a generalization to [8] and [10].

2 Construction of the nonconforming Crouzeix-Raviart element and its anisotropism

For the sake of simplicity, let $\Omega \subset R^2$ be a rectangular domain with boundary $\partial\Omega$ parallel to x-axis or y-axis in the plane, T_h be a family of axiparallel rectangular meshes of Ω which does not need to satisfy the regular condition or quasi-uniform assumption. For any $K \in T_h$, denote the central point of element K by (x_K, y_K), the vertices by $d_i(i = 1, 2, 3, 4)$, the sides by $l_i = \overline{d_i d_{i+1}}(i = 1, 2, 3, 4(mod 4))$, the length of edges parallel to x-axis and y-axis by $2h_x$ and $2h_y$, respectively.

Throughout this paper, we shall use the Sobolev spaces $H^m(\Omega)$, where m is a nonnegative integer. On these spaces we have the norms and seminorms

$$||\mu||_{m,\Omega} = (\sum_{i\leq m} |\mu|^2_{i,\Omega})^{\frac{1}{2}}$$

and

$$|\mu|_{i,\Omega} = (\sum_{|\alpha|=i} |D^\alpha \mu|^2_{0,\Omega})^{\frac{1}{2}}.$$

We will further denote by $H_0^m(\Omega)$ the subspace of functions in $H^m(\Omega)$ that vanish for $D^i f(i \leq m-1, \forall f \in H^m(\Omega))$ on $\Gamma = \partial\Omega$. To further simplify notations we often drop the subscript Ω in the norm when the context is clear.

Let $\widehat{K} = [-1, 1]^2$ be a reference element in ξ-η plane with vertices $\widehat{d}_1 = (-1, -1), \widehat{d}_2 = (1, -1), \widehat{d}_3 = (1, 1)$ and $\widehat{d}_4 = (-1, 1); \widehat{l}_1 = \overline{\widehat{d}_1\widehat{d}_2}, \widehat{l}_2 = \overline{\widehat{d}_2\widehat{d}_3}, \widehat{l}_3 = \overline{\widehat{d}_3\widehat{d}_4}$ and $\widehat{l}_4 = \overline{\widehat{d}_4\widehat{d}_1}$ be the four sides of $\widehat{K}$. The affine transformation $F_K : \widehat{K} \longrightarrow K$ is defined by

$$\begin{cases} x = x_K + h_x\xi \\ y = y_K + h_y\eta. \end{cases} \tag{1}$$