NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 112-130, 2007

# Spectral Element Viscosity Methods for Nonlinear Conservation Laws on the Semi-Infinite Interval

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Received April 23, 2006; Accepted (in revised version) October 29, 2006

**Abstract**

In this paper we propose a spectral element vanishing viscosity (SEVV) method for the conservation laws on the semi-infinite interval. By using a suitable mapping, the problem is first transformed into a modified conservation law in a bounded interval, then the well-known spectral vanishing viscosity technique is generalized to the multi-domain case in order to approximate this transformed equation more efficiently. The construction details and convergence analysis are presented. Under a usual assumption of boundedness of the approximation solutions, it is proven that the solution of the SEVV approximation converges to the unique entropy solution of the conservation laws. A number of numerical tests is carried out to confirm the theoretical results.

**Keywords:** Spectral element methods; spectral vanishing viscosity; conservation laws; unbounded domain.
**Mathematics subject classification:** 35L65, 65M10, 65M15

## 1. Introduction

Spectral methods represent a relatively new approach to the numerical solution of partial differential equations as compared to some more "classical" methods, such as finite difference and finite element methods. Since its appearance (see, e.g., [2, 6]), spectral methods have been applied with success to a broad variety of mathematical equations, and particularly those modelling fluid dynamics. As a matter of fact, the most attractive property of spectral methods may be that when the solution of the problem is infinitely smooth the rate of convergence is exponential (the so-called spectral accuracy). However, when spectral methods are used to solve hyperbolic problems (for example nonlinear conservation laws), there are some difficulties stemming essentially from the fact that hyperbolic problems feature the presence of discontinuous solutions, arising in nonlinear equations, as well as in linear problems with discontinuous initial data. In the classical spectral approximations of the nonlinear conservation laws, the oscillations produced by the Gibbs

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phenomenon often grow up due to the nonlinearities and the calculations fail even if the initial data is smooth. Several approaches have been introduced to overcome this difficulty. The spectral vanishing viscosity (SVV) method, among others, is specially of interest because it allows to stabilize the scheme, while preserving the spectral accuracy. It was first established by Tadmor [17] for the resolution of conservation laws using the Fourier spectral method. The non-periodic case was then considered in the frame of the spectral Legendre approximation in [7, 9–11, 15]. Recently, SVV methods have been also used in the calculation of the incompressible flow of high Reynolds number [18], as well as in large eddy simulations [8, 13]. The analysis of these methods is available only on problems in bounded and single domain.

In this paper we essentially focus on applying the SVV method to conservation laws in a unbounded domain. Precisely, we consider spectral methods for the following problem:

$$\partial_t v(y,t) + \partial_y f(v(y,t)) = 0, \quad (y,t) \in (0,\infty) \times (0,T].$$

Generally, there are three basic ways to construct global approximations to functions defined on the unbounded domain:

(i) truncate the unbounded domain to $[0, L]$, with a suitably large enough $L$;

(ii) employ the Laguerre polynomials or the Laguerre functions to expand the functions;

(iii) map the semi-infinite interval into a finite one and then expand the solution by using Legendre polynomials.

Here we follow the third way: employ a suitable mapping to geometrically transform the unbounded interval into a bounded one so that the classical SVV method can be applied.

The contributions of this work are three folds: Firstly, after transformation, a modified conservation law is obtained. Our convergence analysis shows that the additional term in front of the flux will not present fundamental difficulty in maintaining the efficiency of the classical SVV method for conservation laws. Secondly, in order to improve the efficiency, we construct a spectral method with the domain partition, which is based on the so-called $C^0$ spectral element method. Thirdly, when the multi-domain is concerned, the following point must be fixed: what is the appropriate definition of the SVV operator. We are going to see that due to the weak formulation the global SVV operator can be defined naturally. A detailed convergence analysis and numerical experiments are carried out. It is worthy to mention that with the domain partition, one can expect the corresponding algebraic problem to be better-conditioned because in this case it is not required to use large polynomial degree.

The paper is organized as follows. In Section 2, we formulate the problem and propose a spectral element vanishing viscosity method (SEVV) for conservation laws on the semi-infinite interval. Then, in Section 3 we carry out the convergence analysis. The numerical experiments are presented in Section 4.