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A Solution of Inverse Eigenvalue Problems for Unitary Hessenberg Matrices

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Abstract

Let $H \in \mathbb{C}^{n\times n}$ be an $n \times n$ unitary upper Hessenberg matrix whose subdiagonal elements are all positive. Partition H as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \tag{0.1}$$

where H_{11} is its $k \times k$ leading principal submatrix; H_{22} is the complementary matrix of H_{11}. In this paper, H is constructed uniquely when its eigenvalues and the eigenvalues of $\widehat{H}_{11}$ and $\widehat{H}_{22}$ are known. Here $\widehat{H}_{11}$ and $\widehat{H}_{22}$ are rank-one modifications of H_{11} and H_{22} respectively.

Keywords: Unitary upper Hessenberg matrix; Schur parameters; inverse eigenvalue problem.
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1. Introduction

Let $\mathscr{H}_n$ denote the set of unitary upper Hessenberg matrices of order n with positive subdiagonal elements. It is known that any $H \in \mathscr{H}_n$ can be written uniquely as the products

$$H \doteq H(\gamma_1,\gamma_2,\cdots,\gamma_n) = G_1(\gamma_1)\cdots G_{n-1}(\gamma_{n-1})\widetilde{G}_n(\gamma_n) \tag{1.1}$$

where

$$G_k(\gamma_k) = diag\left[I_{k-1}, \begin{pmatrix} -\gamma_k & \sigma_k \\ \sigma_k & \overline{\gamma_k} \end{pmatrix}, I_{n-k-1}\right], \qquad k = 1,2,\cdots,n-1, \tag{1.2}$$

and

$$\widetilde{G}_n(\gamma_n) = diag[I_{n-1}, -\gamma_n].$$

The parameters $\gamma_k \in \mathbb{C}$, $1 \le k \le n$, are called *reflection coefficients* or *Schur parameters* in signal processing and satisfy $|\gamma_k|^2 + \sigma_k^2 = 1$, $\sigma_k > 0$, $k = 1,\cdots,n-1$, and $|\gamma_n| = 1$. We also refer to (1.1) as *Schur parametric form* of H [7] and to (1.2) as the complex Givens

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matrices. In this paper, I_j denotes the $j \times j$ identity matrix, e_j denotes the j-th column of the identity matrix, and $\lambda(T)$ denotes the spectrums of a square matrix T.

Two kinds of inverse eigenvalue problems for unitary Hessenberg matrices have been considered up to now. One is described in [1] and the methods for constructing a unitary Hessenberg matrix from spectral data are described in [1, 10]. It tells us that $H \in \mathscr{H}_n$ is uniquely determined by its eigenvalues and the eigenvalues of a multiplicative rank-one perturbation of H. Another inverse eigenvalue problem appears in [2] which demonstrates that $H \in \mathscr{H}_n$ can also be determined by its eigenvalues and the eigenvalues of a modified $(n-1) \times (n-1)$ leading principal submatrix of H. All of them are analogous to relevant inverse eigenvalue problems of Jacobi matrices, *i.e.*, real symmetric tridiagonal matrices with positive subdiagonal elements. Recent work by Jiang [9] proves a kind of inverse eigenvalue problem for Jacobi matrices.

Theorem 1.1. [9] *Given two real number sets $\{\lambda_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n-1}$. If there is no common number between $\mu_1, \mu_2, \cdots, \mu_{k-1}$ and $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$, and*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{k-1}} < \lambda_k < \mu_{j_k} < \lambda_{k+1} < \cdots < \mu_{j_{n-1}} < \lambda_n,$$

where $(j_1, \cdots, j_{n-1})$ is a unique permutation of $(1, 2, \cdots, n-1)$, then there exists a unique Jacobi matrix T, such that $\lambda(T) = \{\lambda_i\}_{i=1}^{n}$, $\lambda(T_{1,k-1}) = \{\mu_i\}_{i=1}^{k-1}$, and $\lambda(T_{k+1,n}) = \{\mu_i\}_{i=k}^{n-1}$, where $T_{1,k-1}$ is the $(k-1) \times (k-1)$ leading principal submatrix of T, and $T_{k+1,n}$ is the complementary submatrix of $T_{1,k}$ ($T_{1,k}$ is the $k \times k$ leading principal submatrix of T).

Because the unitary upper Hessenberg matrices with positive subdiagonal elements have rich mathematical structures which are analogous to Jacobi matrices, we propose a new inverse eigenvalue problem for the matrix $H \in \mathscr{H}_n$ similar to Theorem 1.1. That is, if we know all the eigenvalues of H and all the eigenvalues of matrices $\widehat{H}_{11}$ and $\widehat{H}_{22}$, which are rank-one modifications of H_{11} and H_{22} respectively, can we construct the matrix H uniquely? Note that there is a little difference between Theorem 1.1 and our question: we just modify the last column of H_{11} and the first row of H_{22} instead of deleting the k-th row and the k-th column from H.

The paper is organized as follows. In Section 2, using the notation in (1.1), we introduce two modified submatrices $\widehat{H}_{kk}$, $k = 1, 2$. Then the relations of spectral decompositions between H and $\widehat{H}_{kk}$, $k = 1, 2$, are discussed. At the end of this section, a rank-one modification on unitary diagonal matrix, which has the same eigenvalues with H, is obtained. Here the methods we used are analogous to an eigendecomposition in divide and conquer algorithm for unitary eigenproblem (see, e.g., [3,6,8]). In Section 3, we discuss the strictly interlacing properties between the eigenvalues of H and of $\widehat{H}_{11}$ and $\widehat{H}_{22}$ on the assumption that there is no common number between the eigenvalues of $\widehat{H}_{11}$ and $\widehat{H}_{22}$. Then we describe how to construct H from two sets of spectra uniquely, and obtain the main theorem of this paper. In the final section, a numerical algorithm is proposed.