INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 3, Pages 386–406

ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR PARABOLIC PROBLEMS WITH CONCENTRATED DATA

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Abstract. Parabolic equations with unbounded coefficients and even generalized functions (in particular Dirac–delta functions) model large–scale of problems in the heat–mass transfer. This paper provides estimates for the convergence rate of difference scheme in discrete Sobolev like norms, compatible with the smoothness of the differential problems solutions, i.e with the smoothness of the input data.

Key Words. concentrated capacity, Sobolev spaces, generalized solution, difference scheme, rate of convergence.

1. Introduction

The present paper continues the study for convergence of finite difference schemes of the model heat equation with concentrated capacity in [11], [12]. In the heat capacity coefficient the Dirac–delta distribution is involved and as a result, the jump of the heat flow at the interface point is proportional to the time derivative of the temperature. Dynamical boundary conditions correspond to concentrated capacity on the boundary [5], [7], [19]. These problems are nonstandard and the classical analysis is difficult to be applied for error estimates and convergence proof. The finite difference method for parabolic problems with discontinuous data (coefficients, initial and boundary conditions) is based on associated weak solutions [17], [20], [30]. For these problems the most used tool for studying convergence of the difference solutions is the Bramble–Hilbert lemma and its generalizations [4], [6]. The theory of difference scheme convergence rate estimates **compatible** with the smoothness of the differential problem solutions was developed first for elliptic problems in papers of Samarskii, Lazarov and Makarov, cf. the monograph [23]. Further development of this theory is presented in [8], and especially, results for parabolic problems. The basic physical model corresponding to the parabolic problems considered in the present paper is that of heat–transfer, where the process take places in two adjoining bodies at different scale in each body. The diffusion through thin layers, divided the bodies has high specific heat. We consider the limiting case, when the thickness of the layers goes to zero and where the specific heat goes to infinity. The simplest mathematical model of this phenomena is derived in [26] and its further development in [5], [19]. Our aim is to treat these problems as a first order abstract–evolution equation (1), with selfadjoint positive linear operators A, B, defined in Hilbert space H and then to use energy methods

Received by the editors January 1, 2007 and, in revised form, March 22, 2007.
2000 *Mathematics Subject Classification.* 65M15.
The research of the first author was supported by Ministry of Science of Republic of Serbia under project 144005A and Bulgarian National Fund of Sciences under project HS-MI-106/2005, while the research of the second author was supported by Bulgarian National Fund of Sciences under project HS-MI-106/2005.

from the theory of Hilbert spaces. Discrete analysis of appropriate subspaces of the Sobolev spaces are used and yet that allow the discrete operators to be selfadjoint of the space involved. In this first stage we obtain a priori estimates for the discrete solutions. The second important idea of the method consists in constructing the special integral representations of the error of the difference schemes. This allows us by applying imbedding Sobolev's theorems to obtain more accurate estimates. We do not use the Bramble–Hilbert lemma. The remainder of this paper is organized as follows: energy estimates for the solutions of an abstract Cauchy problem for a first order evolution equation and for an operator–difference scheme can be found in the next section. These auxiliary results are used in the next sections for obtaining a priori estimates to the derivation of convergence rate estimates in special discrete Sobolev norms of difference schemes approximations to heat equation with discontinuous coefficients and dynamical conditions of conjugation, i.e in which the time derivative of the solution is involved. We also treat parabolic equations with dynamical boundary condition and elliptic equation with dynamical condition of conjugation. The method proposed here is applied to analogous hyperbolic problem in [13], see also [25]. Energy stability for a class of two-dimensional interface parabolic problems is investigated in [15], while the stability of difference schemes for parabolic equations with dynamical boundary conditions and conditions on conjugation is analyzed in [16]. Two–dimensional elliptic problems in which the Dirac–delta function appears in the lowest coefficients are treated in [9] and [14], while finite-difference approximation for Poisson's equation with a dynamic boundary condition is given in [29]. Convergence of difference schemes on classical solutions for parabolic and hyperbolic equations with dynamical boundary conditions or dynamical conditions of conjugation are studied in [1], [2], [3], [28].

2. Preliminary Results

Let H be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and S – unbounded selfadjoint positive definite linear operator, with domain $D(S)$ dense in H. The product $(u,\, v)_S = (Su,\, v)$ $(u,\ v \in D(S))$ satisfies the inner product axioms. Reinforcing $D(S)$ in the norm $\|u\|_S = (u,\, u)_S^{1/2}$ we obtain a Hilbert space $H_S \subset H$. The inner product $(u,\, v)$ continuously extends to $H_S^* \times H_S$, where H_S^* is the adjoint space for H_S. Operator S extends to mapping $S : H_S \to H_S^*$. There exists unbounded selfadjoint positive definite linear operator $S^{1/2}$, such that $D(S^{1/2}) = H_S$ and $(u,\, v)_S = (Su,\, v) = (S^{1/2}u,\, S^{1/2}v)$ (see [17], [21]). We also define the Sobolev spaces $W_2^s(a,b;\, H)$, $W_2^0(a,b;\, H) = L_2(a,b;\, H)$, of the functions $u = u(t)$ mapping interval $(a,b) \subset R$ into H [17]. Let A and B be unbounded selfadjoint positive definite linear operators, not depending on t, in Hilbert space H, with $D(A)$ – dense in H_B. In general, A and B are noncommutative. We consider an abstract Cauchy problem [20], [30].

$$B\frac{du}{dt} + Au = f(t), \quad 0 < t < T; \qquad u(0) = u_0, \tag{2.1}$$

where u_0 is a given element in H_B, $f(t) \in L_2(0,T;\, H_{A^{-1}})$ – given function and $u(t)$ – unknown function from (0,T) into H_A. Setting in (1) $f(t) = dg(t)/dt$ we get the Cauchy problem

$$B\frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T; \qquad u(0) = u_0. \tag{2.2}$$

The following proposition holds.