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Improved Local Projection for the Generalized Stokes Problem

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Abstract. We analyze pressure stabilized finite element methods for the solution of the generalized Stokes problem and investigate their stability and convergence properties. An important feature of the methods is that the pressure gradient unknowns can be eliminated locally thus leading to a decoupled system of equations. Although the stability of the method has been established, for the homogeneous Stokes equations, the proof given here is based on the existence of a special interpolant with additional orthogonal property with respect to the projection space. This makes it much simpler and more attractive. The resulting stabilized method is shown to lead to optimal rates of convergence for both velocity and pressure approximations.

AMS subject classifications: 65N12, 65N30, 65N15, 76D07

Key words: Generalized Stokes equations, stabilized finite elements, local projection, convergence, error estimates.

1 Introduction

Stabilized finite element methods that circumvent the restrictive $\inf-\sup$ condition have been developed for Stokes-like problems (see, e.g., [4,14,16,19,20]). These residual-based methods represent one class of stabilized methods. They consist in modifying the standard Galerkin formulation by adding mesh-dependent terms, which are weighted residuals of the original differential equations. Although for properly chosen stabilization parameters, these methods are well posed for all velocity and pressure pairs. These methods are sensitive to the choice of the stabilization parameters. Another class of stabilized methods has been derived using Galerkin methods enriched with bubble functions (see, [1,3]). Alternative stabilization techniques based on a continuous penalty method have also been proposed and analyzed in [10,11].

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Recently, local projection methods that seem less sensitive to the choice of parameters and have better local conservation properties were proposed. The stabilization by projecting the pressure gradient has been analyzed in [12]. It was shown that the method is consistent in the sense that a smooth exact solution satisfies the discrete problem. Though the method may seem computationally expensive due to the nonlocal behaviour of the projection, iterative solvers were developed to make the method more efficient ([13]). Alternatively, a two-level approach with a projection onto a discontinuous finite element space of a lower degree defined on a coarser grid has been analyzed in [5,22,23]. In [6,7], low order approximations of the Oseen equations were analyzed.

A drawback of the two-level, from the implementation point of view, is that the added stabilizing term leads to a larger stencil which may not fit the data structure of an available programming code. In [21], stability of local projection methods is proved based on the existence of a special interpolant with additional properties with respect to the projection space. This general approach paves the way for introducing equal order stabilized methods by local projection onto a discontinuous space defined on the same mesh. In this case, the added stabilizing terms do not lead to a larger stencil like the two-level approach.

The main objective of this paper is to analyze the pressure gradient stabilization method for the generalized Stokes problem using the new approach. These kind of problems arise naturally in the time discretization of the unsteady Stokes problem, or the full Navier-Stokes equations by means of an operator splitting technique. Unlike the proof given by [22] and [23], where stability was shown using an inf-sup condition due to [16] and the equivalence of norms on finite dimensional spaces, here, the stability of the pressure-gradient method is proved for arbitrary Q^k-elements, by constructing a special interpolant with additional orthogonal property with respect to the projection space (see, e.g., [24,25]). As a result, optimal rates of convergence are found for the velocity and pressure approximations.

2 Variational formulation

Let Ω be a bounded two-dimensional polygonal region, $f \in L^2(\Omega)$, σ a positive real number, typically,

$$\sigma = \frac{1}{\Delta t},$$

where Δt is the time step in a time discretization procedure, and ν the kinematic viscosity coefficient. Then, the generalized homogeneous Stokes Problem reads: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ satisfying:

$$\begin{cases} \sigma\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$