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MATHEMATICAL FRAMEWORK FOR LATTICE PROBLEMS

BENQI GUO AND IVO BABUŠKA

Dedicated to Professor Max Gunzburger on the occasion of his 60th birthday

Abstract. We propose a mathematical framework to effectively study lattice materials with periodic and non-periodic structures over entire spaces in one, two, and three dimensions. The existence and uniqueness of solutions for periodic lattice problems with absolute terms are proved in discrete Sobolev spaces. By Fourier transform discrete lattice problems are converted to semi-discrete problems for which similar results are establish in semi-discrete Sobolev spaces. For lattice problems without absolute terms, additional conditions are imposed on data for the existence and uniqueness of solutions in discrete energy spaces in one,two and three dimensions. Two concrete examples are analyzed in the proposed mathematical framework. The mathematical framework, methodology and techniques in this paper can be utilized or generalized to non-periodic lattices on entire spaces and boundary value problems on lattices.

Key Words. lattice, cell, multi-scale, periodic structures, grids, absolute term, linear interpolation, Fourier transform.

1. Introduction

Lattice materials are porous materials consisting of periodic cells or non-periodic cells. The cells are composed of rods, or shells, or solid structures. The size of cell is usually small with respect to the size of the body filled with the lattice materials. The lattice materials with simple micro-structures are characterized by a single length scale, for instance, Lattice Block Materials which are developed by JAMCORP corporation. The hierarchic lattice materials have hierarchic multi-scale structures. In either case, we deal with a multi-scale problem. The lattice materials can offer significantly higher strength-to-weight and stiffness-to-weight ratio than their base materials. Hence they can be potentially advantageous in practical engineering applications.

Various micro mechanical models for the lattice materials with periodic and non-periodic structures have been studied for the analysis of the overall properties, crack propagation, etc. There are papers addressing these problems, especially in the mechanics, material science, and physics literatures [8, 12, 17, 19, 21]. For mathematical theory which is related to the problem of the lattice materials we refer to the book [5] and her various papers,e.g. [4, 6, 7]. Recently, asymptotic analysis for periodic lattice problem and multi-scale numerical method based on Fourier transform and homogenization appeared in [13, 14, 15, 20]. In these papers, the scale of cells is assumed so small that asymptotic arguments can be utilized and only problems in presence of absolute term are addressed so that the corresponding bilinear form satisfies the inf-sup condition on a pair of Sobolev spaces. In practical

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applications, these assumptions may not be valid, a substantial adjustment and generalization are needed.

In our paper we focus on periodic lattice materials composed of rods and balls in entire spaces. Such a structure results in a system of difference equations with infinite number of unknowns. We intend to establish a mathematical framework for systematical research on such lattice problems in entire spaces of one, two, and three dimensions. This framework can be used or generalized for lattice materials with complicated micro-structures such as plates and shells, or three dimensional solid structures. The analysis and method developed in this paper can be utilized for boundary value problems on lattices and non-periodic lattice problems, which will be illustrated in a coming paper [9, 10].

For lattice problems with absolute term, the existence and uniqueness of the solutions of variational equation and equilibrium equation are proved in these discrete Sobolev spaces over lattices. The Fourier transform is a powerful tool for studying periodic lattice problems, deriving homogenization results and designing effective computations. The Fourier transform converts discrete lattice problems to semi-discrete problems for which the theorem on existence and uniqueness of solutions is proved in proper semi-discrete Sobolev spaces.

For the lattice problems without absolute terms, we impose additional condition on the data, and substantially modify the discrete Sobolev spaces for the existence and uniqueness of solutions. For proving the results in two and three dimensional lattice problems, we need to utilize the properties of functions in $H^1(R^d), d = 2, 3$, which are attached in Appendix. To utilizing these properties we extend a grid function defined on a lattice to a continuous and piecewise linear function defined over whole space $R^d, d = 2, 3$ based on a proper triangular or tetrahedral partition of R^d, and establish the equivalence of discrete Sobolev norms of grid functions and Sobolev norms of its extension. The techniques of partition and extension can be generalized to non-periodic lattice problems.

The paper is organized as the follows. In Section 2, we first introduce various discrete Sobolev and energy spaces, and prove the existence and uniqueness of the solutions of variational equation for the lattice problems with absolute term. With help of Fourier transform, we convert a fully discrete problem over lattices to a semi-discrete problem over a combination of a bounded domain and the micro structure of the cells. The existence and uniqueness of the solutions for the semi-discrete problems are proved, which are parallel to those for fully discrete problems. In Section 3 we address the lattice problems without absolute terms in the energy spaces for data in the weighted discrete L^2 spaces, which lead to the existence and uniqueness of the solutions. We develop a representation formula for the solutions of the lattice problems in terms of Fourier transform and its inverse in Section 4. We present two examples of lattice problem in the last section, one is a one-dimensional model, another is two-dimensional model. Some concrete formulas for these examples will be derived, which are very helpful to understand lattice problems in general setting. In Appendix, we give some important properties of functions in $H^1(R^d), d = 2, 3$, which are essential to analysis of lattice problems without absolute terms.

2. General Setting in d-dimensions

2.1. Lattice in entire spaces