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ASYMPTOTIC BEHAVIOR OF THE TE AND TM APPROXIMATIONS TO SECOND HARMONIC GENERATION *

Naïma Aïssa
(*Faculté de Mathématiques, USTHB, BP32, El Alia, Bab Ezzouar, 16111, Alger;*
Centre de Mathématiques Appliquées, CNRS UMR 7641 & Ecole Polytechnique, Palaiseau 91128 Cedex, France
Email: aissa@cmapx.polytechnique.fr, n.aissa@yahoo.fr)

Abstract

We consider a Second Harmonic Generation (SHG) problem of an optical signal wave with an optical pump in a medium represented by a smooth bounded domain $\Omega \subset \mathbb{R}^d$, which is assumed to contain a heterogeneous material: a compactly imbedded subdomain $B^r \subset\subset \Omega$ in the shape of a small ball contains a nonlinear material, while $\Omega \setminus \overline{B^r}$ is filled with a linear material. We begin by proving existence and uniqueness of the solution to the TE approximation of SHG for arbitrary bounded susceptibilities, thus improving the result obtained by Bao and Dobson (Eur. J. Appl. Math. 6 (1995), 573-590) under small enough susceptibilities assuption. We then establish an existence and uniqueness result of a solution to the TM approximation problem. In both parts we study the asymptotic behavior of the system as the size of the nonlinear material vanishes: error estimates and asymptotic expansion of the solution are derived for both TE and TM approximations.

Mathematics subject classification: 35Q60, 35B40, 35C20.
Key words: Second harmonic generation, TE Approximation, TM Approximation.

1. Introduction

Franken *et al.* [12] in their second harmonic generation experiment developed a process for generating double frequency laser beams, thus marking the advent of the field of nonlinear optics. Though revolutionary, the theory that underpins the discovery is quite basic: a given medium is subjected to an intense beam of optical pump waves, causing the field in the medium to be polarized nonlinearly. The former and latter process are governed respectively by the constitutive equations and a linear system of Maxwell's equations. For further details, we refer the reader to the very interesting book by Shen [19].

In this paper, we consider a domain Ω which is filled with a heterogeneous material. Inside the domain, a ball-shaped subdomain $B^r \subset\subset \Omega$ of small size, with center x_0 and radius r contains a nonlinear material, while $\Omega \setminus \overline{B^r}$ is filled with a linear material.

We first use the two-dimensional space model introduced in [6] to deal with the TE approximation (*i.e* the diffracted electric fields are assumed to be directed in the vertical direction): we improve the existence and uniqueness result stated in [6] under small enough susceptibilities assumption. Then we study the TM approximation (*i.e.,* the diffracted magnetic and electric fields, with respectively the same and the double frequency as the incident wave, are assumed to be directed in the vertical direction) in the setting of the model proposed in [8] in the three-dimensional case, thereby bypassing the two-dimensional case which presents a technical difficulty for deriving the asymptotic expansion of the solution. More precisely, we prove existence and uniqueness of the solution, and furthermore establish the well-posedness of the problem.

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In both the TE and TM approximations, the behavior of the solution is considered as the size of the nonlinear material vanishes. Moreover, error estimates and asymptotic expansion are derived.

2. The TE Approximation

2.1. Model Problem

Consider the model set in [6]. Throughout the paper, we assume that the medium is non-magnetic and has constant magnetic permeability. For the sake of convenience, the magnetic permeability parameter is set to 1. In addition, we also assume that no external charges nor current are present in the field.

The time-harmonic Maxwell equations which govern second harmonic generation (SHG) take the form

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{i\omega}{c}\mathbf{H}, \ \ \nabla \cdot \mathbf{H} = 0, \\ \nabla \times \mathbf{H} = \frac{i\omega}{c}\mathbf{D}, \ \ \nabla \cdot \mathbf{D} = 0, \end{cases} \tag{1}$$

along with the constitutive equation

$$\mathbf{D} = \varepsilon\ (\mathbf{E} + 4\pi\mathbf{P}), \tag{2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ the magnetic field, $\mathbf{D}$ the electric displacement, $\mathbf{P}$ the polarization field, ε the electric permittivity of the medium, c the speed of light and ω the angular frequency. The physics of SHG may be described as follows: when a plane wave with frequency $\omega = \omega_1$ is projected onto a nonlinear medium, it generates two diffracted waves with respective angular frequencies $\omega = \omega_1$ and $\omega = \omega_2 = 2\omega_1$ because of the interaction between the incident wave and the nonlinear medium. The presence of new frequency components is the most striking difference between nonlinear and linear optics. For most media however, the nonlinear optical effects are so negligible that they may be ignored. To observe nonlinear phenomena in the optical region, one needs high-intensity beams like high-intensity laser ones.

Let us consider the two wave fields $\mathbf{E}(x, \omega_1)$ and $\mathbf{E}(x, \omega_2 = \omega_1 + \omega_1)$. To simplify our notation, we denote $\mathbf{E}(x, \omega_i) = \mathbf{E}(\omega_i)$.

Since second harmonic generation can be considered as a special case of optical mixing [19], the polarization field at frequencies ω_1 and ω_2 respectively are given by [19, p. 68]

$$\begin{cases} \mathbf{P}(w_1) = \chi^{(1)}(\omega_1).\mathbf{E}(\omega_1) + \chi^{(2)}(x, \omega_1) : \mathbf{E}^\star(\omega_1)\mathbf{E}(\omega_2), \\ \mathbf{P}(w_2) = \chi^{(1)}(\omega_2).\mathbf{E}(\omega_2) + \chi^{(2)}(x, \omega_2) : \mathbf{E}(\omega_1)\mathbf{E}(\omega_1), \end{cases}$$

where $\chi^{(1)}$ is the linear susceptibility tensor of the medium, $\chi^{(2)}$ is the second-order nonlinear susceptibility tensor of third rank, that means that, $\chi^{(2)} : \mathbf{EE}$ is a vector whose jth component is $\sum_{k,l=1}^{3} \chi^{(2)}_{jkl}\mathbf{E}_k\mathbf{E}_l$, and $\mathbf{E}^\star$ is the complex conjugate of $\mathbf{E}$. Then the Maxwell equations (1)-(2) yield the following coupled system

$$\begin{cases} \left[\nabla \times (\nabla \times) - \frac{\omega_1^2 d_1}{c^2}\right]\mathbf{E}(\omega_1) = \frac{4\pi\omega_1^2\varepsilon}{c^2}\chi^{(2)}(\omega_1 = -\omega_1 + \omega_2) : \mathbf{E}^\star(\omega_1)\mathbf{E}(\omega_2), \\ \left[\nabla \times (\nabla \times) - \frac{\omega_2^2 d_2}{c^2}\right]\mathbf{E}(\omega_2) = \frac{4\pi\omega_2^2\varepsilon}{c^2}\chi^{(2)}(\omega_2 = \omega_1 + \omega_1) : \mathbf{E}(\omega_1)\mathbf{E}(\omega_1), \end{cases}$$

where $d_i = \varepsilon(1 + 4\pi\chi^{(1)}(\omega_i))$. The medium is said to be linear if $\mathbf{D} = \varepsilon(1 + \chi^{(1)}(\omega))\mathbf{E}$, *i.e.*, $\chi^{(2)}$ vanishes. We assume that all the fields are invariant in the vertical direction. Then the problem can be formulated in two dimensions. We shall also assume that the electric fields at ω_1 and ω_2 are **TE** polarized, which means that

$$\mathbf{E}(\omega_i) = e(\omega_i)\mathbf{u}_3, \ \ i = 1, 2,$$