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A SHIFT-SPLITTING PRECONDITIONER FOR NON-HERMITIAN POSITIVE DEFINITE MATRICES *[1)]

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Abstract

A shift splitting concept is introduced and, correspondingly, a shift-splitting iteration scheme and a shift-splitting preconditioner are presented, for solving the large sparse system of linear equations of which the coefficient matrix is an ill-conditioned non-Hermitian positive definite matrix. The convergence property of the shift-splitting iteration method and the eigenvalue distribution of the shift-splitting preconditioned matrix are discussed in depth, and the best possible choice of the shift is investigated in detail. Numerical computations show that the shift-splitting preconditioner can induce accurate, robust and effective preconditioned Krylov subspace iteration methods for solving the large sparse non-Hermitian positive definite systems of linear equations.

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1. Introduction

Let $\mathbb{C}^n$ represent the complex n-dimensional vector space, and $\mathbb{C}^{n\times n}$ the complex $n \times n$ matrix space. In this paper, we will consider preconditioning the large sparse system of linear equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{C}^n, \tag{1}$$

where A is a large sparse *non-Hermitian positive definite* matrix (i.e., its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite), and b and x are, respectively, the known and the unknown vectors. Here, we have used A^* to denote the conjugate transpose of the matrix A.

The system of linear equations (1) with a non-Hermitian positive definite coefficient matrix A arises in many problems in scientific and engineering computing, see [24, 2, 23]. When a Krylov subspace iteration method is employed to compute an approximation for its solution $x_* = A^{-1}b$, an economical and effective preconditioner is often demanded in order to improve the computational efficiency, the approximate accuracy and the numerical stability of the referred Krylov subspace iteration method, see [2, 23]. There have been many elegant preconditioners presented and studied in the literature in recent years [24, 1, 2, 23, 3, 4, 21], which are cheaply applicable and practically efficient for matrices of specific structures and properties. These preconditioners can be roughly categorized into the incomplete factorizations [18, 17, 5, 2] and the splitting iterations [24, 3, 4, 21]. Essentially, a preconditioner aims to transform the

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original linear system (1) by a suitable linear transformation such that the spectral property of the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is largely improved, and therefore, the convergence speed of the referred Krylov subspace iteration method is considerably accelerated. However, both incomplete factorization and splitting iteration are only applicable and efficient for special classes of matrices, e.g., a diagonally dominant or an irreducibly weakly diagonally dominant matrix. Even for a Hermitian positive definite matrix, its incomplete Cholesky factorization may break down [17]; and for a non-Hermitian positive definite matrix of strong skew-Hermitian part, the splitting iteration may diverge [9].

For the Hermitian positive definite system of linear equations, considering that the conjugate gradient method is quite efficient when its coefficient matrix has tightly clustered spectrum [2, 12, 23], Bai and Zhang [8] recently presented a class of *regularized conjugate gradient* (RCG) method by first shifting and contracting the spectrum of the coefficient matrix, and then approximating the iterates of the regularized iteration sequence by the *conjugate gradient* (CG) iteration [15, 10, 16]. Therefore, the RCG method is actually an inner/outer iteration method [19, 20, 13, 11, 14] with a standard splitting iteration as its outer iteration, and the CG iteration as its inner iteration. The shifted and contracted matrix leads to a linear polynomial preconditioner and the RCG iteration leads to a nonstationary iteration preconditioner for the Hermitian positive definite linear system.

For the non-Hermitian positive definite system of linear equations (1), in this paper we first present a shift splitting for the coefficient matrix A, and then construct a shift-splitting iteration scheme for the linear system (1). Theoretically, this scheme is proved to be convergent unconditionally to the exact solution of the system of linear equations (1). The shift splitting also naturally induces a simple but effective preconditioner for the coefficient matrix A. Moreover, the shift-splitting preconditioning matrix itself can be again approximated by employing an incomplete factorization or a splitting iteration. This leads to a so-called two-level preconditioner for the system of linear equations (1). In actual applications, we can suitably choose the shift in such a way that the induced splitting matrix has reasonably good diagonally dominant property such that its incomplete factorization or splitting iteration is existent, stable, and accurate. Hence, the two-level preconditioner can lead to a highly efficient Krylov subspace iteration method for solving the system of linear equations (1). These results extend and develop those for Hermitian positive definite linear system studied in [8] to non-Hermitian positive definite one.

The organization of the paper is as follows. In Section 2 we introduce the shift splitting concept and the shift-splitting iteration scheme, and discuss their convergent and preconditioning properties. In Section 3 we describe and analyze the two-level preconditioning technique which is defined by adopting a further approximation of the shift-splitting preconditioning matrix. Numerical results are given in Section 4 to show the feasibility and the effectiveness of the shift-splitting and the corresponding two-level preconditioners when they are employed to accelerate the Krylov subspace iteration methods. Finally, in Section 5 we use brief conclusions to end this paper.

2. The Shift-splitting Preconditioner

For a non-Hermitian positive definite matrix $A \in \mathbb{C}^{n\times n}$, we use $\lambda(A)$ to represent its eigenvalue and $\sigma(A)$ its spectrum set, $\beta_l(A)$ and $\beta_u(A)$ the lower and the upper bounds of the real parts of its eigenvalues, and $\gamma_l(A)$ and $\gamma_u(A)$ the lower and the upper bounds of the imaginary parts of its eigenvalues, respectively. That is to say, we have

$$\beta_l(A) \le \Re(\lambda(A)) \le \beta_u(A) \quad \text{and} \quad \gamma_l(A) \le \Im(\lambda(A)) \le \gamma_u(A),$$

where $\Re(\lambda)$ and $\Im(\lambda)$ represent the real and the imaginary parts of the complex λ. Without causing confusion, sometimes we may neglect the matrix A and simply write these bounds as