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AN EXTENDED DOMAIN METHOD FOR OPTIMAL BOUNDARY CONTROL FOR NAVIER-STOKES EQUATIONS

SANDRO MANSERVISI

Abstract. The matching velocity problem for the steady-state Navier-Stokes system is considered. We introduce an extended domain method for solving optimal boundary control problems. The Lagrangian multiplier method is applied to the extended domain with distributed controls and used to determine the optimality system and the control over the boundary of the inner domain. The existence, the differentiability and the optimality system of the optimal control problem are discussed. With this method inflow controls are shown to be numerical reliable over a large admissible control set. Numerical tests for steady-state solutions are presented to prove the effectiveness and robustness of the method for flow matching.

Key Words. optimal boundary control, optimal design, Navier-Stokes equations, velocity matching problem.

1. Introduction

Optimal boundary control problems associated with the Navier-Stokes equations have a wide and important range of applications such as the design of cars, airplanes and jet engines. Despite the fact that this field has been extensively studied, determining the best boundary control or even a simple effective boundary control for a system governed by the Navier-Stokes equations is still a difficult and time consuming task.

Early studies devoted to optimal boundary control problems for the Navier-Stokes equations can be found, for example, in [1, 8, 15, 16]. The optimal control of the Navier-Stokes equations shows many challenges and has been considered by numerous authors (see for exanple [4, 6, 9, 13, 35, 14, 20, 18, 19, 20, 21, 22, 23, 24, 25, 28, 38] and citations therein). The theoretical treatment of optimal boundary problems concerning with questions of existence, regularity of solutions, and differentiability properties is in some extent satisfactory but the numerical implementation, the analysis, and the consistency of discrete approximations still remain fundamental issues. Many results generally lack a coherent first-order necessary condition and often the regularity assumed cannot be used in numerical algorithms. Other papers deal with re-formulations of the problem, mainly to simplified situations with finite dimension controls.

In order to simplify the description of the problem we consider the two-dimensional steady-state incompressible flow of a viscous fluid with Dirichlet boundary conditions in a region Ω with boundary Γ as shown on the left of Figure 1. The velocity

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$\vec{u}$ and the pressure p satisfy the stationary Navier-Stokes system

$$-\nu\triangle\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{h} \qquad \text{in } \Omega \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \qquad \text{in } \Omega \tag{2}$$

along with the Dirichlet boundary conditions

$$\vec{u} = \vec{g} = \begin{cases} 0 & \text{on } \Gamma_1 \\ \vec{g} & \text{on } \Gamma_c\,, \end{cases} \tag{3}$$

where $\vec{h}$ is the given body force. In (1) ν denotes the inverse of the Reynolds number whenever the variables are appropriately nondimensionalized.

Along the uncontrolled part of boundary boundary Γ_1 the velocity vanishes and the function $\vec{g}$ must satisfy the compatibility condition

$$\int_\Gamma \vec{g}\cdot\vec{n}\,ds = 0 \tag{4}$$

where $\vec{n}$ is the unit normal vector along the surface Γ. If some other types of boundary conditions, e.g., natural boundary conditions or outflow boundary conditions, are specified along the left or right or bottom boundaries, the results given in this paper are formally valid but some technical details in the analysis should be carefully revised. There is a substantial literature discussing the set of all possible

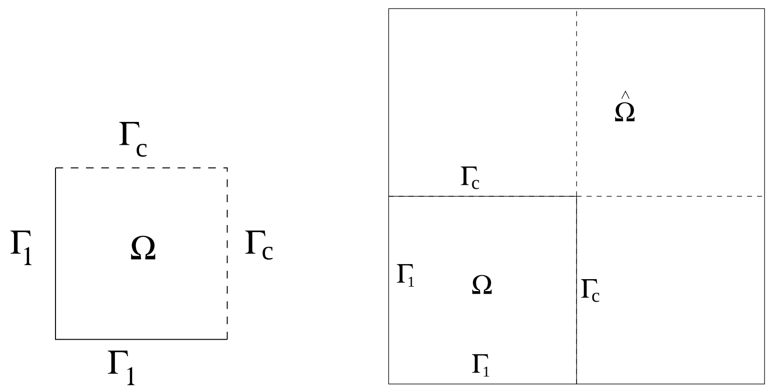

FIGURE 1. The flow domain Ω. Γ_c denotes the part of the boundary whose velocity is to be determined by the optimization process.

boundary controls. Clearly, the function $\vec{g}$ must belong to $H^{1/2}(\Gamma_c)$, the Sobolev space of order 1/2. However $H^{1/2}(\Gamma_c)$ or $H^1(\Gamma_c)$ may not be sufficient to enable one to explicitly derive a first-order necessary condition or implement numerically the boundary control. Thus in general the set of all admissible controls $\vec{g}$ must be restricted to more regular spaces, namely, to belong to $H^{3/2}(\Gamma_c)$.

One could examine several practical objective functionals for determining the boundary controls, e.g., the reduction of the drag due to viscosity or the identification of the velocity at a fixed vertical slit downstream. To fix ideas, we focus on the minimization of the cost functional that leads to matching velocity problems. In literature the steady optimal control problem is formulated by using the following functional (see for example [1, 25])

$$J(\vec{u},\vec{g}) = \frac{1}{2}\int_\Omega (\vec{u}-\widehat{U})^2\,d\vec{x} + \frac{\beta}{2}\int_{\Gamma_c} (\alpha\,\vec{g}_s^{\,2} + \vec{g}^{\,2})\,dx\,, \tag{5}$$