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A TWO-STAGE ALGORITHM OF NUMERICAL EVALUATION OF INTEGRALS IN NUMBER-THEORETIC METHODS*[1)]

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Abstract

To improve the numerical evaluation of integrals in Number-Theoretic Methods, we give a two-stage algorithm. The main idea is that we distribute the points according to the variations of the quadrature on the subdomains to reduce errors. The simulations results are also given.

Key words: Numerical integration, Monte Carlo method, Number-theoretic method.

1. Introduction

The Number-Theoretic Method (NTM) is a special method which represents a combination of number theory and numerical analysis. The widest range of applications and indeed the historical origin of this method is found in numerical integration. Also related problems such as interpolation and the numerical solutions of integral equations and differential equations, optimization and experimental design in statistics can be dealt with successfully. [1–4] give a comprehensive review in bibliographic setting.

In this paper we consider the problem of evaluating integration. Let D be a domain in R^s (s-dimension) and $f(\boldsymbol{X})$ be a continuous function defined on D. We want to calculate the definite integral

$$I(f) = \int_D f(\boldsymbol{X}) d\boldsymbol{X} \tag{1}$$

There are two main approaches in evaluation of $I(f)$. One is Monte Carlo method (MCM) developed by S. Ulam and J. Von Neumann. The basic idea of the Monte Carlo method is to replace an analytic problem by a probabilistic problem with the same solution, and then investigate the latter problem by statistical simulation. For

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simplicity, we consider $D = [0,1]^s$ first. Suppose that $\boldsymbol{X}$ is a random vector which is uniformly distributed on $[0,1]^s$. Then

$$E(f(\boldsymbol{X})) = \int_D f(\boldsymbol{X})d\boldsymbol{X} = I(f)$$

with

$$\sigma(f(\boldsymbol{X})) = \Big[\int_D f^2(\boldsymbol{X})d\boldsymbol{X} - (Ef(\boldsymbol{X}))^2\Big]^{1/2}$$

if they exist. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ be independent samples of $\boldsymbol{X}$ and

$$I(f,n) = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{X}_i) \tag{2}$$

By the strong law of large numbers, $I(f,n)$ converges to $I(f)$ with probability one as $n \to \infty$. Moreover, $I(f,n)$ is approximately normally distributed when n is large by the central limit theorem. Also the law of the iterated logarithm shows that with probability one

$$\lim_{n\to\infty} \sup \sqrt{\frac{n}{2\ln(\ln n)}}\Big|\frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{X}_i) - I(f)\Big| = \sigma^2(f(\boldsymbol{X}))$$

Another approach is the use of the number-theoretic method (NTM). The number-theoretic method for evaluation of the integral is based on the theory of the uniform distribution. Let $P_n = \{\boldsymbol{X}_k, k = 1,2,\cdots,n\}$ be an NT-net on $[0,1]^s$ with low discrepancy (cf. Fand and Wand (1994)). Then we may use

$$I(f,P_n) = \frac{1}{n}\sum_{k=1}^{n} f(\boldsymbol{X}_k) \tag{3}$$

as an approximation for $I(f)$.

Definition. *Let $(n; h_1, h_2, \cdots, h_s)$ be a vector with integral components satisfying $1 \le h_i < n$, $h_i \ne h_j$ $(i \ne j)$, $s < n$ and the greatest common divisors $(n, h_i) = 1$, $i = 1, \cdots, s$. Let*

$$\begin{cases} g_{ki} = kh_i (mod\, n) \\ x_{ki} = (2g_{ki} - 1)/2n \end{cases} \quad k = 1,2,\cdots,n,\ i = 1,2,\cdots,s \tag{4}$$

where we use the usual multiplicative operation module n such that g_{ki} is confined by $1 \le g_{ki} \le n$. Then the set $P_n = \{\boldsymbol{X}_k, k = 1,2,\cdots,n\}$ is called the lattice point set of the generating vector $(n; h_1, h_2, \cdots, h_s)$. If the set P_n has the discrepancy $o(n^{-\frac{1}{2}})$, then the set P_n is called a glp set. It can be seen that x_{ki} defined in (4) can be easily calculated by

$$x_{ki} = \Big\{\frac{2kh_i - 1}{2n}\Big\} \tag{5}$$

where $\{x\}$ stands for the fraction part of x. In one dimension case $P_n = \{(2k-1)/2n, k = 1,2,\cdots,n\}$. The convergence rate of $I(f,P_n)$ can reach $O(n^{-1}(\log n)^s)$ which is