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ANTIPERIODIC WAVELETS*1)

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Abstract

In this paper, we construct the orthogonal wavelet basis in the space of antiperiodic functions by appealing the spline methods. Differing from other results in papers[1,2,3,6,8], here we derive the 3-scale equation, by using this equation we construct some basic functions, those functions can be used to construct different orthonormal basis in some spline function spaces.

1. Preliminary

As we know, many authors have made great efforts in constructing the orthonormal or biorthonormal basis on the whole real line $\mathbb{R}^1$ or on the whole n-dimensional space $\mathbb{R}^n$[4,7], but in many practical problems one needs to construct orthonormal basis on some finite interval with some boundary conditions.

Here we present a method of constructing the antiperiodic orthonormal wavelets basis on the interval $I = [0, 2\pi]$.

Main difficulty in the above problem is the construction of the orthonormal basis of W_{m-1}—the orthogonal complement of V_{m-1} in V_m—the key step is that we have to construct o.n. periodic wavelets $\{A_{\nu,3}^{n,m}\}$ which satisfy 2-scale equations, therefore, we shall adopt some new strategy to construct the o.n. basis of W_{m-1} which differs from [1].

Let n, K be integers, $N \geq 1$, n odd, $n = 2n_0 + 1, 2\pi = Kh, K \geq 2n+2$, h a real number. The point set $\{y_i\}$ are defined as follows

$$y_0 = -\frac{(n+1)}{2}h, \quad y_j = y_0 + jh, \quad j = 1, 2, \cdots \tag{1.1}$$

The B-spline function is defined by

$$B_i^n(x) = (-1)^{n+1}(y_{n+1+i} - y_i)[y_i, \cdots, y_{i+n+1}]_y (x-y)_+^n \tag{1.2}$$

Definition 1.1. $S^{n,m} := \{S | S$ *is a polynomial of degree* n *on each interval* $[jh_m, (j+1)h_m), j \in \mathbb{Z}, S \in C^{n-1}(\mathbb{R}^1)\}$, *where* $h_m = h/3^m, m \geq 0$, m *is an integer.*

Set $g_i^{n,m}(x) = B_i^n(3^m x)$, *then* $\{g_i^{n,m}\}_{i \in \mathbb{Z}}$ *is a basis of* $S^{n,m}$.

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Definition 1.2. $\overset{\circ}{\mathcal{S}}_{n,K(m)} :=$ *{S|S is a polynomial of degree n on each interval* $[jh_m, (j+1)h_m), j = 0, \cdots, K(m)-1; S \in C^{n-1}(I), S^{(j)}(0) = S^{(j)}(2\pi), j = 0, 1, \cdots, n-1\}$. *where* $K(m) = 3^m K$.

$\overset{\circ}{\mathcal{S}}_{n,K(m)}$ is the family of periodic spline functions of degree n and with $K(m)$ knots $\{jh_m\}_{j=0}^{K(m)-1}$ in $[0, 2\pi)$. Set $\widetilde{B}_i^n(x) = B_i^n(x) + B_{i+K}^n(x)$, the system of functions $\{\widetilde{B}_i^n\}_{i=-n_0}^{K-n_0-1}$ constitutes a basis in $\overset{\circ}{\mathcal{S}}_{n,K}$, where $B_i^n(x)$ and $B_{i+K}^n(x)$ are defined as in (1.2), and $K(0) = K$. If we define

$$\widetilde{B}_i^{n,m}(x) := B_i^{n,m}(x) + B_{i+K(m)}^{n,m}(x) = B_i^n(3^m x) + B_{i+K(m)}^n(3^m x) \tag{1.3}$$

then, the system $\{\widetilde{B}_i^{n,m}(x)\}_{i=-n_0}^{K(m)-n_0-1}$ forms a basis of $\overset{\circ}{\mathcal{S}}_{n,K(m)}$.

Definition 1.3. *Given any integer l, there exists unique integer k satisfying*

$$l = k + jK(m), \quad j \in \mathbb{Z} \quad \text{and} \quad -n_0 \le k \le -n_0 - 1 + K(m) \tag{1.4}$$

define

$$\overset{\circ}{B}{}_l^{n,m}(x) = \widetilde{B}_k^{n,m}(x), \quad x \in [0, 2\pi] \ . \tag{1.5}$$

Note here we use the same symbol $\widetilde{B}_i^{n,m}, \overset{\circ}{B}{}_i^{n,m}$ as in [C1], but different in meaning since here $K(m) := 3^m K$ and $h_m := h/3^m$.

From (1.5), we conclude that $\{\overset{\circ}{B}{}_{l+\nu}^{n,m}(x)\}_{\nu=0}^{K(m)-1}$ is a basis in $\overset{\circ}{\mathcal{S}}_{n,K(m)}$. The function $\overset{\circ}{B}{}_l^{n,m}(x)$ can be extended to the whole real axis by periodicity.

We define the inner product of two function f and g on $[0, 2\pi]$ by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$.

Definition 1.4. *Define*

$$A_{k,3}^{n,m}(x) = C_{k,3}^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l k/K(m))\, \overset{\circ}{B}{}_l^{n,m}(x) \tag{1.6}$$

where

$$C_{k,3}^{n,m} = \left\{ \sum_{\nu=0}^{K(m)-1} \left[\exp\left(\frac{2\pi i \nu k}{K(m)}\right)\right] \overset{\circ}{B}{}_l^{2n+1}(0) \right\}^{-\frac{1}{2}} \tag{1.7}$$

Lemma 1.1. $A_{k,3}^{n,m}(x)$ *is defined as in* (1.6)*, then*

$$\langle A_{k,3}^{n,m}(\cdot), A_{j,3}^{n,m}(\cdot)\rangle = \delta_{k,j}, \quad 0 \le k, j \le K(m) - 1. \tag{1.8}$$

Let $V_m := \overset{\circ}{\mathcal{S}}_{n,K(m)}, \{A_{k,3}^{n,m}\}_{k=0}^{K(m)-1}$ *is an o.n. basis in* V_m. *Where* $\delta_{k,j}$ *is the kronecker delta.*