INTERVAL ITERATIVE METHODS UNDER PARTIAL ORDERING (II)*

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Abstract

Many types of nonlinear systems can be solved by using ordered iterative methods. These systems are discussed in [2] in a unified form for five different initial conditions. This paper is a continuation of [2]. Under arbitrary initial conditions, some iterative methods are given, and several theorems for the existence and uniqueness of the solution and convergence of the methods are proved.

§ 1. Introduction

In this paper we consider nonlinear systems

$$\varphi(x)=x, \quad x\in R^n. \tag{1.1}$$

Suppose there are f_i: $R^{r_i}\times R^{s_i}\to R$, such that

$$\varphi_i(x)=f_i(A_ix, B_ix), \quad i=1, 2, \cdots, n \tag{1.2}$$

where $A_i\in R^{r_i\times n}$, $B_i\in R^{s_i\times n}$, $0\leqslant r_i$, $s_i\leqslant n$, $f_i(A_ix, B_iy)$ are isotone in x and antitone in y when the latter are comparable, that is, as $x\leqslant x'$, $y\geqslant y'$, $x\leqslant y$ or $x\geqslant y$, $x'\leqslant y'$ or $x'\geqslant y'$, we have

$$f_i(A_ix, B_iy)\leqslant f_i(A_ix', B_iy'), \quad i=1, 2, \cdots, n.$$

Most of the functions discussed in [1] (13.2—13.5) can be written in form of (1.2). For simplicity, we suppose $A=A_i$, $B=B_i$, $i=1, 2, \cdots, n$, and consider

$$\varphi(x)=f(Ax, Bx)=x. \tag{1.3}$$

Clearly, (1.3) and (1.2) are equivalent.

We define some notation as follows:

$[\underline{x}, \bar{x}]=\{u|\underline{x}\leqslant u\leqslant\bar{x}\}$ is an n-dimensional interval vector, $\underline{x}$, $\bar{x}\in R^n$.

$N=\{1, 2, \cdots, n\}$.

$F[\underline{x}, \bar{x}]=[f(A\underline{x}, B\bar{x}), f(A\bar{x}, B\underline{x})]$.

$L_w[\underline{x}, \bar{x}]=[\underline{x}+w(f(A\underline{x}, B\bar{x})-\underline{x}), \bar{x}+w(f(A\bar{x}, B\underline{x})-\bar{x})]$ where $w\in R$, $w>1$.

$R[\underline{x}, \bar{x}]=[\underline{x}+Q(f(A\bar{x}, B\underline{x})-\bar{x}), \bar{x}+Q(f(A\underline{x}, B\bar{x})-\underline{x})]$ where Q is a nonnegative and nonsingular $n\times n$ matrix.

We will use the following lemmas.

Lemma 1. (1) *F is an inclusion monotonic interval extension of $\varphi(x)=f(Ax, Bx)$.*

(2) *If there exists $1>\beta>0$ such that*

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$$f(Ax, By)-f(Ax', By')\geqslant\beta(x-x'),\ y'\geqslant y,\ x\geqslant x'$$

for all comparable x, y *and* x', y', *let* $1/(1-\beta)\geqslant w>1$. *Then* L_w *is an inclusion monotonic interval extension of* $l(x)=x+w(f(Ax, Bx)-x)$.

(3) *If there exists* $P\in R^{n\times n}$, *such that*

$$f(Ax, By)-f(Ax', By')\leqslant P(y'-y)+(x-x'),\quad y'\geqslant y,\ x\geqslant x'$$

for all comparable x, y *and* x', y', *let* Q *be a nonnegative, nonsingular, left subinverse of* P. *Then* R *is an inclusion monotonic interval extension of* $r(x)=x+Q(f(Ax, Bx)-x)$.

Lemma 1 is a conclusion of several theorems in [2].

Lemma 2. *Let* f: $R^n\to R^n$ *be continuously differentiable on* R^n. *Assume that* $f'(x)-I$ *is nonsingular and* $\|(f'(x)-I)^{-1}\|\leqslant\beta<\infty$ *for all* $x\in R^n$. *Then for any fixed* $x^0\in R^n$, *there exists a unique continuously differentiable mapping* x: $[0, 1]\to R^n$ *such that*

$$g(x(t),t)=x(t),$$
$$x'(t)=(f'(x)-I)^{-1}(x^0-\hat{x}),\quad t\in[0, 1],\ x(0)=x^0$$

where $$g(x, t)=tf(x)+(1-t)d(x),\ d(x)=f(x)-\hat{x}+x^0,\ f(x^0)=\hat{x}.$$

§ 2. Algorithms and Convergence

Algorithm 1. Define initial interval $[\underline{x}^0, \overline{x}^0]$.

1. If $F[\underline{x}^k, \overline{x}^k]\cap R[\underline{x}^k, \overline{x}^k]\cap[\underline{x}^k, \overline{x}^k]=\varnothing$, then the algorithm is stopped.
2. $[\underline{x}^{k+1}, \overline{x}^{k+1}]=F[\underline{x}^k, \overline{x}^k]\cap R[\underline{x}^k, \overline{x}^k]\cap[\underline{x}^k, \overline{x}^k]$.

Theorem 1. *Suppose that* $f(Ax, By)$ *is continuous in* x, $y\in[\underline{x}^0, \overline{x}^0]$ *and there are* $1\geqslant r>0$, $P=\mathrm{diag}(p_1, p_2, \cdots, p_n)>0$, *such that*

$$f(Ax, By)-f(Ax', By')\leqslant P(y'-y)+(x-x'), \tag{2.1}$$
$$|f(Ax, Bx')-x|+|f(Ax', Bx)-x'|\geqslant r(x-x') \tag{2.2}$$

for all comparable x, y *and* x', y', $y\leqslant y'$, $x\geqslant x'$, x, y, x', $y'\in[\underline{x}^0, \overline{x}^0]$. *Then there exists a unique solution of* (1.3) *in* $[\underline{x}^0, \overline{x}^0]$ *if and only if Algorithm* 1 *can be continued indefinitely. In this case it yields a sequence* $\{[\underline{x}^k, \overline{x}^k]\}$ *for which*

(1) $$[\underline{x}^{k+1}, \overline{x}^{k+1}]\subseteq[\underline{x}^k,\overline{x}^k], \tag{2.3}$$
$$\overline{x}^{k+1}-\underline{x}^{k+1}\leqslant t(\overline{x}^k-\underline{x}^k) \tag{2.4}$$

where $0\leqslant t=\max\limits_{1\leqslant i\leqslant n}\{1-q_ir/(q_i+1)\}$, $Q=\mathrm{diag}(q_1, q_2, \cdots, q_n)>0$, $QP\leqslant I$,

$$\lim_{k\to\infty}\overline{x}^k=\lim_{k\to\infty}\underline{x}^k=x';$$

(2) *there exists a unique solution* $x^*=x'$ *of* (1.3).

Proof. If there exists a solution x^* of (1.3) in $[\underline{x}^0, \overline{x}^0]$, then by Lemma 1 we have

$$x^*\in F[\underline{x}^0, \overline{x}^0],\ x^*\in R[\underline{x}^0, \overline{x}^0].$$

From Algorithm 1, we have $x^*\in[\underline{x}', \overline{x}']$. We can easily show by induction that

$$x^*\in[\underline{x}^k, \overline{x}^k].$$