Journal of Computational Mathematics
Vol.31, No.5, 2013, 470–487.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1304-m4288

TWO-GRID CHARACTERISTIC FINITE VOLUME METHODS FOR NONLINEAR PARABOLIC PROBLEMS*

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Abstract

In this work, two-grid characteristic finite volume schemes for the nonlinear parabolic problem are considered. In our algorithms, the diffusion term is discretized by the finite volume method, while the temporal differentiation and advection terms are treated by the characteristic scheme. Under some conditions about the coefficients and exact solution, optimal error estimates for the numerical solution are obtained. Furthermore, the two-grid characteristic finite volume methods involve solving a nonlinear equation on coarse mesh with mesh size H, a large linear problem for the Oseen two-grid characteristic finite volume method on a fine mesh with mesh size $h=\mathcal{O}(H^2)$ or a large linear problem for the Newton two-grid characteristic finite volume method on a fine mesh with mesh size $h=\mathcal{O}(|\log h|^{1/2}H^3)$. These methods we studied provide the same convergence rate as that of the characteristic finite volume method, which involves solving one large nonlinear problem on a fine mesh with mesh size h. Some numerical results are presented to demonstrate the efficiency of the proposed methods.

Mathematics subject classification: 35Q55, 65N30, 76D05.
Key words: Two-grid, Characteristic finite volume method, Nonlinear parabolic problem, Error estimate, Numerical example.

1. Introduction

Many processes in science and engineering are described by the parabolic equations, for instance, the processes of fluid dynamics, hydrology and environmental protection [20, 25]. There have been extensive works devoted to linear parabolic problems see, e.g., the monographs [30]. For nonlinear cases, we mention only [9, 26] and the references therein.

In this paper, we consider the following nonlinear parabolic problem in $\mathbb{R}^2$:

$$\begin{cases} u_t+\nabla\cdot(a(u)\nabla u)+\mathbf{b}(u)\nabla u=f(u), & \text{in } \Omega\times(0,T], \\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T], \\ u(\cdot,0)=u_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where Ω is a bounded convex polygonal domain with a sufficiently smooth boundary $\partial\Omega$, $\nabla=(\partial/\partial x_1,\partial/\partial x_2)^T$, and $\mathbf{b}(u)=(b_1(u),b_2(u))^T$ is a vector function. We define a bounded set on $\mathbb{R}^2$ as

$$G=\{u:|u|\le K_0\}, \tag{1.2}$$

where K_0 is a positive constant.

Supposing the coefficients of problem (1.1) satisfy the following conditions:

* Received December 21, 2012 / Revised version received February 25, 2013 / Accepted April 16, 2013 / Published online August 27, 2013 /

$(C_1):$ $a(u)$ and $f(u)$ are Lipschitz continuous with respect to u, i.e.

$$|g(u) - g(v)| \leq L|u - v|, \quad \text{for } \forall\ u, v \in G, \tag{1.3a}$$

where L is a Lipschitz constant related to K_0, $g(u)$ can take $a(u)$ or $f(u)$.

$(C_2):$ $a(u)$ is a bounded smooth function with positive upper and lower bounds,

$$0 < a_* \leq a(u) \leq a^*, \quad \text{for } \forall\ u \in G. \tag{1.3b}$$

$(C_3):$ $f(u)$ is a given real-valued function on Ω and there is a constant M such that

$$|f'(u)| + |f''(u)| \leq M, \quad \text{for } \forall\ u \in G, \tag{1.3c}$$

where $f'(u) = \frac{df(u)}{du}$. Under the conditions above, problem (1.1) admits a unique solution in a certain Sobolev space (see [30]).

Finite volume method (FVM) as one of numerical discretization techniques has been widely employed to solve the fluid dynamics problems in recent years (see [12, 23] and the references therein). It is developed as an attempt to use finite element ideas in the finite difference setting. The basic idea is to approximate the discrete fluxes of a partial differential equation using the finite element procedure based on volumes or control volumes, so FVM is also called box scheme, general difference method et al. (see [1,3,19]). Finite volume method has many advantages that belong to finite difference method or finite element method, such as, it is easy to set up and implement, conserve mass locally and it also can treat the complicated geometry and general boundary conditions flexibility. However, the analysis of FVM lags far behind that of finite element and finite difference methods, we can refer to the literature [10, 11, 31] for more recent developments about the finite volume method.

The modified method of characteristic (MMOC) was first proposed by Douglas and Russell for the convection-diffusion equations in [8]. After then, a lot of works have been reported about this method. For instance, Russell considered the nonlinear coupled systems in [27], Süli studied the Navier-Stokes equations in [29]. The MMOC is based on the approximation of the material derivative term, that is, the time derivative term plus the convection term, and this scheme works well for convection dominant problem (see [35] and the reference therein).

On the other hand, two-grid method is an efficient numerical scheme for partial differential equations based on two spaces with different mesh sizes. This kind of discretization technique for linear and nonlinear elliptic PDEs was first introduced by Xu in [32, 33]. After then, two-grid method has been studied by many researchers, for example, Dawson et al. considered the nonlinear parabolic equations by using the finite element or finite difference methods in [6, 7], respectively. Marion and Xu [24] applied it to the evolution equations. For the Navier-Stokes equations, we can refer to [15–18, 22]. Recently, Bi and Ginting in [2] combined the two-grid method and the finite volume method for linear and nonlinear elliptic problems.

In this paper, we devote ourselves to the study of two-grid characteristic finite volume method (CFVM) for nonlinear parabolic problem. By introducing an elliptic projection, optimal error estimates of numerical solution are established. Another important novel ingredient of this work is the convergence analysis of the approximate solution in two-grid schemes. We prove that the initial approximation u_H^n of the nonlinear problem is determined on the coarse mesh. Then the fine mesh approximation u_h^{no} or u_h^{nn} is obtained by solving a large linear problem for the Oseen two-grid CFVM on a fine mesh with mesh size $h = \mathcal{O}(H^2)$ or a large linear problem for the Newton two-grid CFVM on a fine mesh with mesh size $h = \mathcal{O}(|\log h|^{1/2}H^3)$, respectively.