Journal of Computational Mathematics
Vol.32, No.2, 2014, 195–204.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1401-m4385

ON L^2 ERROR ESTIMATE FOR WEAK GALERKIN FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS*

Fuzheng Gao
School of Mathematics, Shandong University, Jinan, Shandong 250100, China
Email: fzgao@sdu.edu.cn
Lin Mu
Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
Email: linmu@math.msu.edu

Abstract

A weak Galerkin finite element method with stabilization term, which is symmetric, positive definite and parameter free, was proposed to solve parabolic equations by using weakly defined gradient operators over discontinuous functions. In this paper, we derive the optimal order error estimate in L^2 norm based on dual argument. Numerical experiment is conducted to confirm the theoretical results.

Mathematics subject classification: 65M15, 65M60.
Key words: WG-FEMs, discrete weak gradient, parabolic problem, error estimate.

1. Introduction

We consider in this paper the approximation of a parabolic problem on a bounded domain $\Omega \subset R^2$ of the form

$$u_t - \nabla \cdot (a\nabla u) = f, \quad x \in \Omega, \quad 0 < t \leq T, \tag{1.1a}$$

$$u = u^0, \qquad x \in \Omega, \quad t = 0, \tag{1.1b}$$

with homogenous Dirichlet boundary condition, where u_t is the time partial derivative of $u(x,t)$; $a(x)$ is an uniformly positive on $\bar{\Omega}$ and $a(x)$, $f(x,t)$ and $u^0(x)$ are assumed to be sufficiently smooth. Since the 1950s, scientists have formulated time-stepping procedures to numerically approximate the solutions of such problems.

Numerical methods for such parabolic problems can be classified as two categories. The first category consists of finite difference methods that use difference quotient to replace differential quotient and the other refers to as finite element methods, see, e.g., [3, 5, 6, 11–13, 17] and references in.

The WG-FEMs, which was first introduced by Wang and Ye [15] for solving the second order elliptic problems, are newly developed FEMs. The novel idea of WG-FEMs is to introduce weak functions and weak derivatives, and allows the use of totally discontinuous piecewise polynomials in the finite element procedure. Later, The WG -FEMs were studied from implementation point of view in [7] and applied to solve the Helmholtz problem with high wave numbers in [9].

A WG-FEM was introduced and analyzed for parabolic equations based on a discrete weak gradient arising from local RT [10]. Due to the use of RT elements, the WG finite element

* Received April 27, 2013 / Revised version received November 11, 2013 / Accepted January 15, 2014 / Published online March 31, 2014 /

formulation of [4] was limited to classical finite element partitions of triangles ($d = 2$) or tetrahedra ($d = 3$). In our previous work, we presented a WG-FEM with stabilization term for a parabolic equation. This method is symmetric, positive definite and parameter free, and allows the use of partitions with arbitrary polygons in two dimensions, or polyhedra in three dimensions with certain shape regularity. Optimal convergence rate in H^1 norm and suboptimal convergence rate in L^2 norm for the WG approximation are derived. The objective of this paper is to derive an optimal order error estimate in L^2 norm based on dual argument technique for the solution of the WG-FEM.

The paper is organized as follows. Section 1 is introduction. In Section 2, we define weak gradient and present semi-discrete and full-discrete WG-FEM for problem (1.1). In Section 3, we establish the optimal order error estimates in L^2-norm to the WG-FEM for the parabolic problem based on dual argument. Finally, in Section 4 we give some numerical examples to verify the theory.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted as those in [1,2].

2. A Weak Galerkin Finite Element Method

The variational form to (1.1) is seeking $u = u(x,t) \in L^2(0,T;H_0^1(\Omega))$, such that

$$(u_t, v) + a(u,v) = (f,v), \qquad \forall\, v \in H_0^1(\Omega), \quad t > 0, \tag{2.1a}$$
$$u(x,0) = u^0(x), \qquad x \in \Omega, \tag{2.1b}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$ and $a(\cdot,\cdot)$ is defined in (2.2).

$$a(v,w) = \int_\Omega a\nabla v \cdot \nabla w \mathrm{d}x. \tag{2.2}$$

It is well known that the solution to (2.1) is called generalized solution of (1.1).

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions [14]. Define $(u,v)_T = \int_T uv\mathrm{d}x$ and $\langle u,v\rangle_{\partial T} = \int_{\partial T} uv\mathrm{d}s$. We introduce a trial function space V_h, which is called weak Galerkin finite element space, as follows

$$V_h := \Big\{ v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_k(e), e \subset \partial T, \forall T \in \mathcal{T}_h \Big\}, \tag{2.3}$$

where T^0 and ∂T denote the interior and boundary of element $T \in \mathcal{T}_h$ respectively. Let $P_k(T^0)$ and $P_k(\partial T)$ be the sets of polynomials on T^0 and ∂T with degree no more than k respectively. v_0 represents the value of v on T^0 and v_b represents that of v on ∂T, respectively. We define V_h^0 as a subspace of V_h with zero boundary value, i.e.,

$$V_h^0 := \Big\{ v = \{v_0, v_b\} \in V_h, v_b \mid_{\partial T \bigcap \partial\Omega} = 0, \forall T \in \mathcal{T}_h \Big\}. \tag{2.4}$$

For each $v = \{v_0, v_b\} \in V_h$, we define the weak discrete gradient $\nabla_w v \in [P_{k-1}(T)]^2$ of v on each element T by the equation as:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q) + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall q \in [P_{k-1}(T)]^2. \tag{2.5}$$