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ERROR ANALYSIS OF THE SECOND-ORDER SERENDIPITY VIRTUAL ELEMENT METHOD FOR SEMILINEAR PSEUDO-PARABOLIC EQUATIONS ON CURVED DOMAINS*

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Abstract

The second-order serendipity virtual element method is studied for the semilinear pseudo-parabolic equations on curved domains in this paper. Nonhomogeneous Dirichlet boundary conditions are taken into account, the existence and uniqueness are investigated for the weak solution of the nonhomogeneous initial-boundary value problem. The Nitsche-based projection method is adopted to impose the boundary conditions in a weak way. The interpolation operator is used to deal with the nonlinear term. The Crank-Nicolson scheme is employed to discretize the temporal variable. There are two main features of the proposed scheme: (i) the internal degrees of freedom are avoided no matter what type of mesh is utilized, and (ii) the Jacobian is simple to calculate when Newton's iteration method is applied to solve the fully discrete scheme. The error estimates are established for the discrete schemes and the theoretical results are illustrated through some numerical examples.

Mathematics subject classification: 65M15, 65M60.
Key words: Semilinear pseudo-parabolic equation, Serendipity virtual element method, Projection method, Curved domain.

1. Introduction

Pseudo-parabolic equations are a vital class of mathematical physics equations and can describe a huge amount of physical evolution processes, including non-steady infiltration in fissured rocks [5], the two-temperature theory in thermodynamics [31], phase separation by spinodal decomposition [26] and so forth. In [1], a more detailed survey on the applications of pseudo-parabolic equations is provided.

The focus of this work is the following semilinear pseudo-parabolic equation with nonhomogeneous initial-boundary value conditions:

$$a(\boldsymbol{x})u_t - \nabla \cdot \big(b(\boldsymbol{x})\nabla(u_t + u)\big) + c(u) = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1.1a}$$

$$u = g(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \tag{1.1b}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1c}$$

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where Ω is a convex bounded open subset of $\mathbb{R}^2$ and a finite number of curves $\{(\partial\Omega)_i\}_{i=1}^{N_\Omega}$ constitute its boundary $\partial\Omega$. Each curve $(\partial\Omega)_i$ is assumed to be sufficiently smooth and the boundary $\partial\Omega$ is supposed to be Lipschitz. T denotes the finite terminal time. The source function f, boundary value g and initial value u_0 are given data. The coefficients a and b depend only on $\boldsymbol{x}, c(u)$ is the nonlinear term.

Due to the complexity of the shape of the domain Ω as well as the presence of nonlinear term $c(u)$, it is difficult or even impossible to give the exact solution to (1.1) in an explicit way. Therefore, efficient and accurate numerical methods should be considered. Some typical numerical schemes for linear pseudo-parabolic equations include finite difference schemes [16], finite volume element methods [38], finite element methods [36] and mixed finite element methods [23]. For nonlinear pseudo-parabolic equations, various numerical methods, including characteristic finite element methods [19], conforming and nonconforming finite element methods [27,33] and discontinuous Galerkin methods [29, 37], have been developed. A more comprehensive survey of numerical methods for various types of pseudo-parabolic equations can be found in [1]. In recent years, numerical methods that can deal with polygonal or polyhedral meshes have become important issues in the field of scientific computing, and such methods have been used to numerically solve pseudo-parabolic equations, including weak Galerkin methods [17], hybrid high-order methods [34] and virtual element methods [35].

The above-mentioned virtual element method (VEM) can be deemed as an extension of the finite element method towards meshes with general polygonal or polyhedral elements, and it has been used to numerically approximate a wide range of nonlinear initial-boundary value problems. In the framework of VEM, the common strategy to deal with nonlinear terms is to use L^2-projection operator, and this idea has already been applied to semilinear parabolic problem [2], Swift-Hohenberg equation [15], nonlocal model [4], nonlinear Schrödinger equation [22] and so forth. Recently, the idea of using interpolation operator to deal with nonlinear terms was proposed in [18]. In this idea, the features of the serendipity virtual element method (SVEM) are fully utilized, and a new way of numerically solving nonlinear evolution equations is provided in VEM framework.

SVEM [6] is a novel variant of VEM and its aim is to decrease the amount of internal-moment degrees of freedom. We focus on the second-order SVEM in this paper. The first motivation of our interest is that internal degrees of freedom are completely avoided without the need to consider the relationship between the degree of polynomials adopted in SVEM and the shape of the mesh elements. Indeed, as stated in [6,18], for higher order SVEM and certain types of meshes, some additional internal degrees of freedom may be needed. The second motivation is that when we adopt the idea in [18] to approximate nonlinear term $c(u)$ in (1.1) by interpolation operator and solve the nonlinear system by Newton's iteration, the calculation of Jacobian is convenient. It is well known that Newton's iteration is of second-order convergence, so it is often used to solve nonlinear problems. However, if we use the L^2-projection operator to deal with nonlinear terms as in [2], the calculation of Jacobian is complex and time-consuming. The main reason is that the L^2-projection operator involves the integral on the mesh elements, which makes the form of Jacobian complicated.

Iso-parametric finite elements [21] are popular for partial differential equations on curved domains. This kind of methods relies on the reference element technique, which is not available in VEM or SVEM due to the use of general polygonal or polyhedral meshes [30]. Thus, the iso-parametric idea cannot be easily generalized to VEM or SVEM. Here, we will use the Nitsche-based projection method, which was first proposed in [9] and then extended into the framework of VEM in [7,8]. The boundary conditions are imposed by the Nitsche-based projection method which was first proposed in [9] and then extended into the framework of VEM in [7,8]. To implement the Nitsche-based projection method, we need to know the gradient of functions in virtual element space. But this information is not explicitly known in VEM or SVEM. In this work, for the gradient of functions in the second-order serendipity virtual element space, we replace it with its L^2-projection. Based on trace inequalities, inverse inequalities and some other lemmas, we prove that when the penalty parameter is sufficiently large, the bilinear form provided by the Nitsche-based projection method is continuous and coercive. It is worth noting that, when $\partial\Omega$ is not curved, the Nitsche-based projection method in this work is essentially the same as the Nitsche's method in [32]. However, the variational method in [32] cannot be easily generalized to this work. The reason lies in the fact that VEM or SVEM does not satisfy the Galerkin orthogonality, which is the key tool in the variational method. Therefore, we employ the classical method based on the energy projection for error analysis.

The present paper is structured as follows. The existence and uniqueness of the weak solution of (1.1) are briefly analyzed in Section 2. We present in Section 3 some basic settings of the second-order SVEM and describe the construction of the numerical schemes (including semi-discrete and fully discrete schemes). Error estimates for discrete schemes in regard to L^2-norm and an energy norm are derived in Section 4. Some numerical examples confirming the error analysis are reported in Section 5. Finally, some conclusions are given in Section 6.

2. Existence and Uniqueness of the Weak Solution

Based on the Galerkin method, we analyse the existence and uniqueness of the weak solution of (1.1) in this section.

2.1. Some notations

For indices $s \geq 1$ and $p \geq 1$, the seminorm and norm in Sobolev space $W^{s,p}(\omega)$ are represented by $|\cdot|_{s,p,\omega}$ and $\|\cdot\|_{s,p,\omega}$, respectively. Herein, $\omega \subset \mathbb{R}^2$ is a bounded domain. When $p = 2, W^{s,2}(\omega)$ is written as $H^s(\omega)$, and $|\cdot|_{s,\omega}$ and $\|\cdot\|_{s,\omega}$ are used for the seminorm and norm in $H^s(\omega)$, respectively. $H^{-1}(\Omega)$ is used to denote the dual space of the zero boundary value space $H_0^1(\Omega)$. $\|\cdot\|_\omega$ and $(\cdot\,,\cdot)_\omega$ are used to represent the L^2-norm and inner product in $L^2(\omega)$, respectively. The space of the traces of the $H^s(\omega)$-functions is denoted as $H^{s-1/2}(\partial\omega)$. We also adopt the standard definitions for the Bochner space $L^q[0,\mathfrak{T};H]$ with norm $\|\cdot\|_{L^q[0,\mathfrak{T};H]}$, where H is a Hilbert space, $\mathfrak{T}$ is a positive real number and the index $q \geq 1$.

2.2. Existence and uniqueness

To carry out the analysis in this subsection, we make the following assumption.

Assumption 2.1. *The coefficients and data in* (1.1) *satisfy*

(A1) a and b belong to $L^\infty(\Omega)$ and it is assumed that

$$a_\star \leq a(\boldsymbol{x}) \leq a^\star, \quad b_\star \leq b(\boldsymbol{x}) \leq b^\star, \quad \forall\, \boldsymbol{x} \in \overline{\Omega}, \tag{2.1}$$

where the constants $a_\star, a^\star, b_\star$ and $b^\star$ are all positive.

(A2) g and g_t belong to $H^{1/2}(\partial\Omega)$ for any $t \in (0,T)$, $f \in L^2[0,T;L^2(\Omega)], u_0 \in H^1(\Omega)$.

(A3) $c(u)$ is Lipschitz continuous in respect of u, for any $u \in \mathbb{R}$.

By the assumption (A2), it is well-known that for any $t \in [0,T]$, there is a function $G \in L^2[0,T;H^1(\Omega)]$ satisfying $G|_{\partial\Omega} = g$ and $G_t \in L^2[0,T;H^{-1}(\Omega)]$, see [14]. Thus by setting $\breve{u} = u - G$, we can transform (1.1) into the following problem:

$$a(\boldsymbol{x})\breve{u}_t - \nabla \cdot \big(b(\boldsymbol{x})\nabla(\breve{u}_t + \breve{u})\big) + c(\breve{u} + G) = \check{f}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T], \tag{2.2a}$$
$$\breve{u} = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \tag{2.2b}$$
$$\breve{u}(\boldsymbol{x},0) = \breve{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{2.2c}$$

where
$$\check{f} = f - aG_t + \nabla \cdot \big(b\nabla(G_t + G)\big), \quad \breve{u}_0 = u_0 - G(0).$$
We remark here that we need to assume $G_t \in L^2[0,T;H^1(\Omega)]$ additionally to let (2.2a) make sense.

Let us now define two bilinear forms

$$A(\breve{v},\breve{w}) = \big(a(\boldsymbol{x})\breve{v},\breve{w}\big)_\Omega, \qquad \forall\, \breve{v},\breve{w} \in L^2(\Omega),$$
$$B(\breve{v},\breve{w}) = \big(b(\boldsymbol{x})\nabla\breve{v},\nabla\breve{w}\big)_\Omega, \quad \forall\, \breve{v},\breve{w} \in H^1(\Omega).$$

Then, it is easy to obtain the following coercivity and continuity of $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$:

$$|A(\breve{v},\breve{w})| \le a^\star\|\breve{v}\|_\Omega\|\breve{w}\|_\Omega, \quad a_\star\|\breve{v}\|_\Omega^2 \le A(\breve{v},\breve{v}), \tag{2.3}$$
$$|B(\breve{v},\breve{w})| \le b^\star|\breve{v}|_{1,\Omega}|\breve{w}|_{1,\Omega}, \quad b_\star|\breve{v}|_{1,\Omega}^2 \le B(\breve{v},\breve{v}). \tag{2.4}$$

The variational form of (2.2) is given by finding $\breve{u} \in L^\infty[0,T;H_0^1(\Omega)]$ with $\breve{u}_t \in L^2[0,T;H_0^1(\Omega)]$ such that for a.e. $t \in (0,T)$,

$$A(\breve{u}_t,\breve{v}) + B(\breve{u}_t,\breve{v}) + B(\breve{u},\breve{v}) + \big(c(\breve{u}+G),\breve{v}\big)_\Omega = (\check{f},\breve{v})_\Omega, \quad \forall\, \breve{v} \in H_0^1(\Omega), \tag{2.5}$$
$$\breve{u}(0) = \breve{u}_0. \tag{2.6}$$

Then the weak solution for the original problem (1.1) is defined as $u = \breve{u} + G$.

Theorem 2.1. *Suppose $c(G) \in L^2[0,T;L^2(\Omega)]$ and Assumption* 2.1 *holds, then the problem* (1.1) *has a unique weak solution u such that*

$$u \in L^\infty\big[0,T;H_0^1(\Omega)\big] \cup L^2[0,T;H^1(\Omega)], \quad u_t \in L^2[0,T;H^1(\Omega)]$$

with the stability estimate

$$\begin{aligned}
&\|u\|_{L^\infty[0,t;H^1(\Omega)]} + \|u_t\|_{L^2[0,t;H^1(\Omega)]} \\
&\le \check{C}\big(\|u_0\|_{1,\Omega} + \|G(0)\|_{1,\Omega} + \|f\|_{L^2[0,t;L^2(\Omega)]} \\
&\qquad + \|G_t\|_{L^2[0,t;H^1(\Omega)]} + \|G\|_{L^2[0,t;H^1(\Omega)]} + \|c(G)\|_{L^2[0,t;L^2(\Omega)]}\big),
\end{aligned} \tag{2.7}$$

where $\check{C}$ is a positive constant independent of u and u_t.

Proof. The following steps comprise the proof.

Step 1. Galerkin approximation. Let $\{\check{w}_k\}_{k=1}^{\infty}$ be the orthogonal basis of $H_0^1(\Omega)$. It is well known that $\{\sqrt{\check{\lambda}_k}\check{w}_k\}_{k=1}^{\infty}$ constitute an orthonormal basis for $L^2(\Omega)$, where $\check{\lambda}_k$ is the k-th eigenvalue with respect to the operator determined by the bilinear form $B(\cdot\,,\cdot)$, see [25, Lemma 5]. Construct an approximate solution $\check{u}_m$ for (2.5) and (2.6) of the form

$$\check{u}_m = \sum_{k=1}^{m} \alpha_m^k(t)\check{w}_k$$

such that for $k = 1, 2, \ldots, m$,

$$A(\check{u}_{m,t}, \check{w}_k) + B(\check{u}_{m,t}, \check{w}_k) + B(\check{u}_m, \check{w}_k) + \big(c(\check{u}_m + G), \check{w}_k\big)_\Omega = (\check{f}, \check{w}_k)_\Omega, \tag{2.8}$$

$$\check{u}_m(0) = \check{u}_{m0}, \tag{2.9}$$

where $\check{u}_{m0}$ is a function in $H_0^1(\Omega)$ which strongly converges to $\check{u}_0$ when $m \to \infty$.

It is obvious that (2.8) and (2.9) form an ordinary differential equation system with regard to $\{\alpha_m^k(t)\}_{k=1}^m$. The coercivity and continuity of $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ together with the Lipschitz continuity of c give the existence of a solution $\check{u}_m$ which satisfies (2.8) and (2.9), see [3, Theorem 4.2.2].

Step 2. A priori estimates. Multiplying the k-th equation of (2.8) by $\alpha_m^k(t)$, summing up for $k = 1, 2, \ldots, m$ and integrating from 0 to t, we find that

$$\begin{aligned}
&A\big(\check{u}_m(t), \check{u}_m(t)\big) + B\big(\check{u}_m(t), \check{u}_m(t)\big) \\
&= A\big(\check{u}_m(0), \check{u}_m(0)\big) + B\big(\check{u}_m(0), \check{u}_m(0)\big) + 2\int_0^t \big(\check{f}, \check{u}_m(t)\big)_\Omega \mathrm{d}t \\
&\quad - 2\int_0^t B\big(\check{u}_m(t), \check{u}_m(t)\big)\mathrm{d}t - 2\int_0^t \big(c\big(\check{u}_m(t) + G\big), \check{u}_m(t)\big)_\Omega \mathrm{d}t.
\end{aligned}$$

Then by the coercivity and continuity of $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$, we have

$$\begin{aligned}
\|\check{u}_m(t)\|_{1,\Omega}^2 \le C_1\bigg(&\|\check{u}_m(0)\|_{1,\Omega}^2 + \int_0^t \big(\check{f}, \check{u}_m(t)\big)_\Omega \mathrm{d}t + \int_0^t |\check{u}_m(t)|_{1,\Omega}^2 \mathrm{d}t \\
&+ \int_0^t \big(c\big(\check{u}_m(t) + G\big), \check{u}_m(t)\big)_\Omega \mathrm{d}t\bigg),
\end{aligned} \tag{2.10}$$

where C_1 is a positive constant that only depends on $a_\star, a^\star, b_\star$ and $b^\star$. From (2.9), we know that

$$\|\check{u}_m(0)\|_{1,\Omega}^2 \le C_2\|\check{u}_0\|_{1,\Omega}^2 \tag{2.11}$$

with a positive constant C_2 independent of m. By the Cauchy-Schwarz inequality and Green's formula, it holds

$$\begin{aligned}
\int_0^t \big(\check{f}, \check{u}_m(t)\big)_\Omega \mathrm{d}t &= \int_0^t \Big(\big(f, \check{u}_m(t)\big)_\Omega - \big(aG_t, \check{u}_m(t)\big)_\Omega - \big(b\nabla(G_t + G), \nabla\check{u}_m(t)\big)_\Omega\Big)\mathrm{d}t \\
&\le \int_0^t \big(\|f\|_\Omega + a^\star\|G_t\|_\Omega + b^\star|G_t + G|_{1,\Omega}\big)\|\check{u}_m(t)\|_{1,\Omega}\mathrm{d}t \\
&\le C_3\int_0^t \big(\|f\|_\Omega + a^\star\|G_t\|_\Omega + b^\star|G_t + G|_{1,\Omega}\big)^2\mathrm{d}t + \int_0^t \|\check{u}_m(t)\|_{1,\Omega}^2\mathrm{d}t,
\end{aligned} \tag{2.12}$$

in which the positive constant C_3 is irrelevant to m. According to the Lipschitz continuity of c, we obtain

$$\begin{aligned}
&\int_0^t \big(c(\check{u}_m(t)+G), \check{u}_m(t)\big)_\Omega \mathrm{d}t \\
&= \int_0^t \big(c(\check{u}_m(t)+G) - c(G), \check{u}_m(t)\big)_\Omega \mathrm{d}t \\
&\quad + \int_0^t \big(c(G), \check{u}_m(t)\big)_\Omega \mathrm{d}t \\
&\leq L_c \int_0^t \|\check{u}_m(t)\|_{1,\Omega}^2 \mathrm{d}t + C_4 \int_0^t \|c(G)\|_\Omega^2 \mathrm{d}t, \qquad (2.13)
\end{aligned}$$

where L_c denotes the Lipschitz constant for c and C_4 is a positive constant that only depends on L_c. Therefore, using (2.11)-(2.13), we arrive at

$$\begin{aligned}
\|\check{u}_m(t)\|_{1,\Omega}^2 \leq{}& C_1\bigg(C_2\|\check{u}_0\|_{1,\Omega}^2 + C_3\int_0^t \big(\|f\|_\Omega + a^\star\|G_t\|_\Omega + b^\star|G_t+G|_{1,\Omega}\big)^2 \mathrm{d}t + C_4\int_0^t \|c(G)\|_\Omega^2 \mathrm{d}t\bigg) \\
&+ (2C_1 + C_1L_c)\int_0^t \|\check{u}_m(t)\|_{1,\Omega}^2 \mathrm{d}t.
\end{aligned}$$

Applying further the continuous Grönwall's lemma [28] gives

$$\begin{aligned}
\|\check{u}_m(t)\|_{1,\Omega}^2 &\leq e^{(2C_1+C_1L_c)t} C_1\bigg(C_2\|\check{u}_0\|_{1,\Omega}^2 + C_3\int_0^t \big(\|f\|_\Omega + a^\star\|G_t\|_\Omega + b^\star|G_t+G|_{1,\Omega}\big)^2 \mathrm{d}t \\
&\qquad + C_4\int_0^t \|c(G)\|_\Omega^2 \mathrm{d}t\bigg) \\
&\leq \check{C}_1\bigg(\|\check{u}_0\|_{1,\Omega}^2 + \int_0^t \big(\|f\|_\Omega^2 + \|G_t\|_{1,\Omega}^2 + \|G\|_{1,\Omega}^2 + \|c(G)\|_\Omega^2\big)\mathrm{d}t\bigg) \qquad (2.14)
\end{aligned}$$

with a positive constant $\check{C}_1$ independent of m. The Eq. (2.14) gives the boundedness of $\{\check{u}_m\}_{m=1}^\infty$ in $L^\infty[0,T;H_0^1(\Omega)]$.

Similarly, multiplying the k-th equation of (2.8) by $(\alpha_m^k(t))'$, summing up for $k=1,2,\ldots,m$, and integrating from 0 to t, it holds

$$\begin{aligned}
&\int_0^t \big(A(\check{u}_{m,t}, \check{u}_{m,t}) + B(\check{u}_{m,t}, \check{u}_{m,t})\big)\mathrm{d}t \\
&\leq B\big(\check{u}_m(0), \check{u}_m(0)\big) + \int_0^t \big(\check{f}, \check{u}_{m,t}\big)_\Omega \mathrm{d}t - \int_0^t \big(c(\check{u}_m+G), \check{u}_{m,t}\big)_\Omega \mathrm{d}t,
\end{aligned}$$

then by similar analysis as in (2.11)-(2.13), we have

$$\int_0^t \|\check{u}_{m,t}\|_{1,\Omega}^2 \mathrm{d}t \leq \check{C}_2\left(\|\check{u}_0\|_{1,\Omega}^2 + \int_0^t \big(\|f\|_\Omega^2 + \|G_t\|_{1,\Omega}^2 + \|G\|_{1,\Omega}^2 + \|c(G)\|_\Omega^2\big)\mathrm{d}t\right), \qquad (2.15)$$

where the estimate (2.14) is used and $\check{C}_2$ is a positive constant independent of m. Thus, $\{\check{u}_{m,t}\}_{m=1}^\infty$ is bounded in $L^2[0,T;H_0^1(\Omega)]$.

Step 3. Existence. The estimates (2.14) and (2.15) give the existence of sub-sequences $\{\check{u}_{m_l}\}_{l=1}^\infty, \{\check{u}_{m_k,t}\}_{k=1}^\infty$, which satisfy that $\check{u}_{m_l} \to \check{u}$ weak-star in $L^\infty[0,T;H_0^1(\Omega)]$ and $\check{u}_{m_k,t} \to \check{u}_t$

in $L^2[0,T;H_0^1(\Omega)]$ weakly. By the compactness lemma (see [3, Lemma A.15.1]) and a similar argument as [25, Theorem 6], we deduce that the weak solution $\check{u}$ for (2.2) exists with $\check{u} \in L^\infty[0,T;H_0^1(\Omega)]$ and $\check{u}_t \in L^2[0,T;H_0^1(\Omega)]$. Furthermore, the existence for weak solution of (1.1) follows from the relation $u = \check{u} + G$.

Step 4. Uniqueness. Assume that $u_1 = \check{u}_1 + G$ and $u_2 = \check{u}_2 + G$ are two weak solutions for (1.1). Then we see that $\check{e} = u_1 - u_2 = \check{u}_1 - \check{u}_2$ satisfies that for any $t \in (0,T)$,

$$A(\check{e}_t,\check{v}) + B(\check{e}_t,\check{v}) + B(\check{e},\check{v}) + \big(c(\check{u}_1+G) - c(\check{u}_2+G),\check{v}\big)_\Omega = 0, \quad \forall\, \check{v} \in H_0^1(\Omega), \tag{2.16}$$

$$\check{e}(0) = 0. \tag{2.17}$$

Taking $\check{v} = \check{e}$ in (2.16) and integrating on $(0,t)$, we have

$$\begin{aligned} &A\big(\check{e}(t),\check{e}(t)\big) + B\big(\check{e}(t),\check{e}(t)\big) \\ &= A\big(\check{e}(0),\check{e}(0)\big) + B\big(\check{e}(0),\check{e}(0)\big) - 2\int_0^t B(\check{e},\check{e})\mathrm{d}t \\ &\quad - 2\int_0^t \big(c(\check{u}_1+G) - c(\check{u}_2+G),\check{e}\big)_\Omega \mathrm{d}t. \end{aligned}$$

Then by (2.17), the Lipschitz continuity of c, the coercivity and continuity of $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$, we get

$$\|\check{e}(t)\|_{1,\Omega}^2 \le \hat{C}\int_0^t \|e(t)\|_{1,\Omega}^2 \mathrm{d}t,$$

in which $\hat{C}$ is a positive constant that only depends on $a_\star, a^\star, b_\star$ and $b^\star$. The continuous Grönwall's lemma gives

$$\|\check{e}(t)\|_{1,\Omega}^2 \le 0,$$

thus $\check{e} = 0$ and it is shown that the weak solution for (1.1) is unique.

Step 5. Stability. Taking $\check{v}$ in (2.5) as $\check{u}$ and employing the similar analysis as in (2.11)-(2.14), we arrive at

$$\begin{aligned} \|\check{u}(t)\|_{1,\Omega} \le \check{C}_1\big(&\|u_0\|_{1,\Omega} + \|G(0)\|_{1,\Omega} + \|f\|_{L^2[0,t;L^2(\Omega)]} + \|G_t\|_{L^2[0,t;H^1(\Omega)]} \\ &+ \|G\|_{L^2[0,t;H^1(\Omega)]} + \|c(G)\|_{L^2[0,t;L^2(\Omega)]}\big). \end{aligned} \tag{2.18}$$

Taking the place of $\check{v}$ with $\check{u}_t$ in (2.5), we can similarly obtain

$$\begin{aligned} \|\check{u}_t\|_{L^2[0,t;H^1(\Omega)]} \le \check{C}_2\big(&\|u_0\|_{1,\Omega} + \|G(0)\|_{1,\Omega} + \|f\|_{L^2[0,t;L^2(\Omega)]} + \|G_t\|_{L^2[0,t;H^1(\Omega)]} \\ &+ \|G\|_{L^2[0,t;H^1(\Omega)]} + \|c(G)\|_{L^2[0,t;L^2(\Omega)]}\big). \end{aligned} \tag{2.19}$$

Then the stability estimate (2.7) can be derived by (2.18), (2.19) and the triangular inequality. The proof is complete. □

3. Numerical Schemes

In this section, we introduce the second-order serendipity virtual element space, construct some discrete bilinear forms, propose semi-discrete and fully discrete schemes based on them.

3.1. Second-order serendipity virtual element space

For the curved domain Ω, in order to approximate it, we introduce a sequence of polygonal domains Ω_h whose vertices are assumed to be located on $\partial\Omega$. Herein, the subscript h $(0 < h < 1)$ is used to show how close Ω_h is to Ω. The smaller the value of h is, the closer Ω_h and Ω are. Convexity of Ω implies that Ω_h is also convex and $\Omega_h \subset \Omega$. For each Ω_h, we decompose it with a polygonal mesh $\mathcal{T}_{h^\star}$ where the subscript $h^\star$ represents the size of the polygonal mesh, i.e. $h^\star := \max_{E\in\mathcal{T}_{h^\star}} h_E$ (h_E is the diameter of the polygon E). For the sake of simplicity, we set $h^\star = h$. We assume that the mesh $\mathcal{T}_h$ satisfies some shape regularity, see [6,18]. Taking the circle domain as an example, we show Ω, Ω_h and $\mathcal{T}_h$ in Fig. 3.1.

For any point $\boldsymbol{x} \in \partial\Omega_h$, which is the boundary of Ω_h, it is assumed that there exists a nonnegative function $\rho(\boldsymbol{x})$ satisfying

$$\boldsymbol{x} + \rho(\boldsymbol{x})\boldsymbol{n} \in \partial\Omega,$$

where $\boldsymbol{n}$ typifies the outward unit normal on $\partial\Omega_h$. From [7–9], we know that

$$\rho(\boldsymbol{x}) \le C_\Omega h^2, \tag{3.1}$$

in which the positive constant C_Ω does not depend on h.

Following [6], for any element $E \in \mathcal{T}_h$, we introduce an auxiliary space

$$\widetilde{V}(E) := \left\{ v \in H^1(E) : \Delta v \in \mathbb{P}_2(E), v|_{\partial E} \in C^0(\partial E), v|_e \in \mathbb{P}_2(e), \forall\, e \subset \partial E \right\},$$

where $C^0(\partial E)$ is the continuous function space on ∂E and $\mathbb{P}_m(\omega)$ denotes the space of polynomials of degree m on ω $(\omega = E, e)$ with the integer $m \ge 0$.

Let the number of edges of E be denoted as N_E^e. It is well known that the number of vertices of E is also equal to N_E^e. Define an operator $\mathcal{S}_E : \widetilde{V}(E) \to \mathbb{R}^{2N_E^e}$ such that

$$[\mathcal{S}_E v]_i = \begin{cases} v(\mathcal{V}_i), & \text{if} \quad 1 \le i \le N_E^e, \\ v\big(\mathcal{V}^e_{i-N_E^e}\big), & \text{if} \quad N_E^e + 1 \le i \le 2N_E^e, \end{cases} \tag{3.2}$$

where $\mathcal{V}_j$ $(1 \le j \le N_E^e)$ is the j-th vertex of E and $\mathcal{V}_j^e$ $(1 \le j \le N_E^e)$ is the midpoint of j-th edge of E. Employ the symbol $(\cdot\,,\cdot)_{\mathbb{R}^{2N_E^e}}$ to represent the Euclidean scalar product in $\mathbb{R}^{2N_E^e}$, then we can define the operator $\Pi_{\mathcal{S}}^E : \widetilde{V}(E) \to \mathbb{P}_2(E)$ as

$$\big(\mathcal{S}_E\big(\Pi_{\mathcal{S}}^E v\big), \mathcal{S}_E p\big)_{\mathbb{R}^{2N_E^e}} = (\mathcal{S}_E v, \mathcal{S}_E p)_{\mathbb{R}^{2N_E^e}}, \quad \forall\, p \in \mathbb{P}_2(E).$$

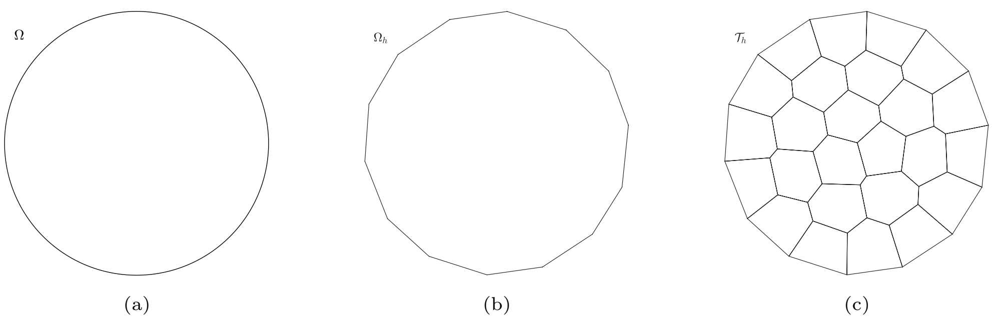

Fig. 3.1. Curved domain Ω (a), polygonal domain Ω_h (b), and polygonal mesh $\mathcal{T}_h$ (c).

Utilizing the operator $\Pi_{\mathcal{S}}^E$, the local serendipity virtual element space $V(E)$ is defined as

$$V(E) := \left\{v_h \in \widetilde{V}(E) : (v_h, p)_E = \left(\Pi_{\mathcal{S}}^E v_h, p\right)_E, \forall\, p \in \mathbb{P}_2(E)\right\}.$$

Take the local degrees of freedom for $V(E)$ as

(D1) The values of v_h at the N_E^e vertexes of E.

(D2) The value of v_h at the midpoint of e, for any $e \subset \partial E$.

In [6], it is proven that (D1) and (D2) are unisolvent for $V(E)$ and it is easy to check that $\Pi_{\mathcal{S}}^E$ is computable using (D1) and (D2). Recall the standard L^2-projection operators Π_0^E and Π_1^E for scalar and vector, respectively, and their definitions are as follows:

$$(w, p)_E = \left(\Pi_0^E w, p\right)_E, \quad \forall\, w \in L^2(E), \qquad \forall\, p \in \mathbb{P}_2(E), \tag{3.3}$$

$$(\boldsymbol{w}, \boldsymbol{p})_E = \left(\Pi_1^E \boldsymbol{w}, \boldsymbol{p}\right)_E, \quad \forall\, \boldsymbol{w} \in \left[L^2(E)\right]^2, \quad \forall\, \boldsymbol{p} \in [\mathbb{P}_1(E)]^2. \tag{3.4}$$

From the definition of $V(E)$, we observe that for functions in $V(E)$, the L^2-projection operator $\Pi_0^E = \Pi_{\mathcal{S}}^E$, and for the gradient of functions in $V(E)$, the L^2-projection operator Π_1^E can be computed by $\Pi_{\mathcal{S}}^E$. Some properties of Π_0^E and Π_1^E are recalled in the following lemma.

Lemma 3.1 ([11]). *If a function $v \in H^3(E)$ for any $E \in \mathcal{T}_h$, we have*

$$\left\|\Pi_0^E v\right\|_E \leq \|v\|_E, \tag{3.5}$$

$$\left\|\Pi_1^E \nabla v\right\|_E \leq \|\nabla v\|_E, \tag{3.6}$$

$$\left\|\nabla v - \Pi_1^E \nabla v\right\|_E \leq \widetilde{C_*} h_E^{s-1} |v|_{s,E}, \qquad s \in \{2,3\}, \tag{3.7}$$

$$\left\|v - \Pi_0^E v\right\|_E + h_E \left|v - \Pi_0^E v\right|_{1,E} + h_E^2 \left|v - \Pi_0^E v\right|_{2,E} \leq C_* h_E^s |v|_{s,E}, \quad s \in \{2,3\}, \tag{3.8}$$

where $\widetilde{C_*}$ and C_* are two positive constants irrelevant to h_E.

We construct the second-order global serendipity virtual element space $V_{\mathcal{S}}$ as

$$V_{\mathcal{S}} := \left\{v_h \in H^1(\Omega_h) : v_h|_E \in V(E),\ \forall\, E \in \mathcal{T}_h\right\}, \tag{3.9}$$

and for $V_{\mathcal{S}}$, the global degrees of freedom are defined by the coupling of the local ones (D1) and (D2).

Let the amount of vertices and edges of $\mathcal{T}_h$ be N^v and N^e, respectively. Obviously, the amount of global degrees of freedom $N_{\mathcal{S}}$ equals $N^e + N^v$. We use $\texttt{dof}_i$ to represent the i-th global degree of freedom, then the interpolation operator $\mathcal{I}_{\mathcal{S}}$ can be defined as

$$\texttt{dof}_i(\psi) = \texttt{dof}_i(\mathcal{I}_{\mathcal{S}}\psi), \quad i = 1, 2, \ldots, N_{\mathcal{S}}, \tag{3.10}$$

where ψ is a regular function. The following approximation properties hold true for $\mathcal{I}_{\mathcal{S}}$.

Lemma 3.2 ([13, 18]). *If a function $\psi \in H^3(E), \forall\, E \in \mathcal{T}_h$, we have*

$$\|\psi - \mathcal{I}_{\mathcal{S}}\psi\|_E + h_E |\psi - \mathcal{I}_{\mathcal{S}}\psi|_{1,E} \leq C_{\mathrm{I}} h_E^s |\psi|_{s,E}, \quad s \in \{2,3\}, \tag{3.11}$$

where C_{I} is a positive constant irrelevant to h_E.

3.2. Discrete bilinear forms

For any v_h and w_h in $V_{\mathcal{S}}$, we first construct a discrete bilinear form $\mathcal{A}_h(\cdot\,,\cdot)$ to approximate the inner product $(av_h, w_h)_{\Omega_h}$. For any $E \in \mathcal{T}_h$, define a local bilinear form $\mathcal{A}_h^E(\cdot\,,\cdot)$ on $V(E) \times V(E)$ with the form

$$\mathcal{A}_h^E(v_h, w_h) := \left(a\Pi_0^E v_h, \Pi_0^E w_h\right)_E + \bar{a}h_E^2 \left(\mathcal{S}_E\left(v_h - \Pi_0^E v_h\right), \mathcal{S}_E\left(w_h - \Pi_0^E w_h\right)\right)_{\mathbb{R}^{2N_E^e}},$$

where

$$\bar{a} = \frac{1}{|E|}\int_E a(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

and the operators $\mathcal{S}_E$ and Π_0^E are defined in (3.2) and (3.3), respectively. Then $\mathcal{A}_h(\cdot\,,\cdot)$ is constructed as

$$\mathcal{A}_h(v_h, w_h) := \sum_{E\in\mathcal{T}_h} \mathcal{A}_h^E(v_h, w_h), \quad \forall\, v_h, w_h \in V_{\mathcal{S}}.$$

We next construct a discrete bilinear form $\mathcal{B}_h(\cdot\,,\cdot)$ to approximate $(b\nabla v_h, \nabla w_h)_{\Omega_h}$. Define a local bilinear form $\mathcal{B}_h^E(\cdot\,,\cdot)$ on $V(E) \times V(E)$ as

$$\mathcal{B}_h^E(v_h, w_h) := \left(b\Pi_1^E \nabla v_h, \Pi_1^E \nabla w_h\right)_E + \bar{b}\left(\mathcal{S}_E\left(v_h - \Pi_0^E v_h\right), \mathcal{S}_E\left(w_h - \Pi_0^E w_h\right)\right)_{\mathbb{R}^{2N_E^e}},$$

where

$$\bar{b} = \frac{1}{|E|}\int_E b(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

and Π_1^E is defined in (3.4). We then construct $\mathcal{B}_h(\cdot\,,\cdot)$ as

$$\mathcal{B}_h(v_h, w_h) := \sum_{E\in\mathcal{T}_h} \mathcal{B}_h^E(v_h, w_h), \quad \forall\, v_h, w_h \in V_{\mathcal{S}}.$$

To deal with the nonlinear term $c(u)$, we further need to define a discrete bilinear form $\mathcal{C}_h(\cdot\,,\cdot)$ which is used to approximate the inner product $(v_h, w_h)_{\Omega_h}$. Similarly, define a local bilinear form $\mathcal{C}_h^E(\cdot\,,\cdot)$ on $V(E) \times V(E)$ with the form

$$\mathcal{C}_h^E(v_h, w_h) := \left(\Pi_0^E v_h, \Pi_0^E w_h\right)_E + h_E^2\left(\mathcal{S}_E\left(v_h - \Pi_0^E v_h\right), \mathcal{S}_E\left(w_h - \Pi_0^E w_h\right)\right)_{\mathbb{R}^{2N_E^e}},$$

and $\mathcal{C}_h(\cdot\,,\cdot)$ is constructed as

$$\mathcal{C}_h(v_h, w_h) := \sum_{E\in\mathcal{T}_h} \mathcal{C}_h^E(v_h, w_h), \quad \forall\, v_h, w_h \in V_{\mathcal{S}}.$$

Introduce a discrete inner product $\langle\cdot\,,\cdot\rangle_{\partial\Omega_h}$ on $L^2(\partial\Omega_h)$ with the form

$$\langle z_1, z_2\rangle_{\partial\Omega_h} = \sum_{e\in\mathcal{E}_h^b}\int_e z_1 z_2 \mathrm{d}s,$$

where $\mathcal{E}_h^b$ represents the set of boundary edges of $\mathcal{T}_h$. We further define a piecewise operator Π_1 which is defined as

$$(\Pi_1 \boldsymbol{v})|_E = \Pi_1^E(\boldsymbol{v}|_E), \quad \forall\, \boldsymbol{v} \in \left[L^2(\Omega_h)\right]^2, \quad \forall\, E \in \mathcal{T}_h.$$

We now construct a discrete bilinear form $\mathcal{N}_h(\cdot\,,\cdot)$ according to the Nitsche-based projection method [7,8]. Based on $\mathcal{B}_h(\cdot\,,\cdot)$, we define $\mathcal{N}_h(\cdot\,,\cdot)$ as

$$\mathcal{N}_h(v_h, w_h) := \mathcal{B}_h(v_h, w_h) - \langle b(\Pi_1 \nabla v_h)\cdot \boldsymbol{n}, w_h\rangle_{\partial\Omega_h}$$

$$- \langle v_h, b(\Pi_1 \nabla w_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + \gamma \big\langle v_h, \bar{h}^{-1} w_h \big\rangle_{\partial\Omega_h}$$
$$- \langle \rho(\Pi_1 \nabla v_h) \cdot \boldsymbol{n}, b(\Pi_1 \nabla w_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + \gamma \big\langle \rho(\Pi_1 \nabla v_h) \cdot \boldsymbol{n}, \bar{h}^{-1} w_h \big\rangle_{\partial\Omega_h},$$

where γ is a penalty parameter and $\bar{h}$ is a piecewise function which is defined as $\bar{h}|_e = h_e$ for any $e \in \mathcal{E}_h^b$ (h_e denotes the length of e).

Based on Lemmas 3.3 and 3.4, we will analyze the coercivity and continuity of $\mathcal{A}_h(\cdot\,,\cdot)$, $\mathcal{B}_h(\cdot\,,\cdot)$, $\mathcal{C}_h(\cdot\,,\cdot)$ and $\mathcal{N}_h(\cdot\,,\cdot)$ in Theorem 3.1. Before that, introduce an energy norm $\|\cdot\|_{\mathcal{N}}$ for $V_{\mathcal{S}}$ with the form

$$\|v_h\|_{\mathcal{N}}^2 \coloneqq |v_h|_{1,\Omega_h}^2 + |v_h|_{\partial\Omega_h}^2, \quad \forall\, v_h \in V_{\mathcal{S}}, \tag{3.12}$$

where

$$|v_h|_{\partial\Omega_h}^2 \coloneqq \sum_{e\in\mathcal{E}_h^b} h_e^{-1} \|v_h\|_e^2.$$

We find that if $\|v_h\|_{\mathcal{N}} = 0$, then v_h equals a constant in Ω_h and $v_h = 0$ on $\partial\Omega_h$. Hence $v_h \equiv 0$ in Ω_h. Obviously, $\|\cdot\|_{\mathcal{N}}$ defines a norm on $V_{\mathcal{S}}$.

Lemma 3.3 ([13, 18]). *For any $E \in \mathcal{T}_h$ and any $v_h \in V(E)$, we have the following inverse inequality and norm equivalence:*

$$\|\nabla v_h\|_E \le C_{\text{in}} h_E^{-1} \|v_h\|_E, \tag{3.13}$$

$$C_{\text{eq}}^{\text{d}} h_E^2 \big(\mathcal{S}_E(v_h), \mathcal{S}_E(v_h)\big)_{\mathbb{R}^{2N_E^e}} \le \|v_h\|_E^2 \le C_{\text{eq}}^{\text{u}} h_E^2 \big(\mathcal{S}_E(v_h), \mathcal{S}_E(v_h)\big)_{\mathbb{R}^{2N_E^e}}, \tag{3.14}$$

where the positive constants $C_{\text{in}}, C_{\text{eq}}^{\text{d}}$ and C_{eq}^{u} are irrelevant to h_E.

Lemma 3.4 ([10]). *We have the following trace inequality for any $E \in \mathcal{T}_h$:*

$$\|\boldsymbol{p}\|_{\partial E} \le C_{\text{dt}} h_E^{-\frac{1}{2}} \|\boldsymbol{p}\|_E, \quad \forall\, \boldsymbol{p} \in [\mathbb{P}_1(E)]^2, \tag{3.15}$$

in which the positive constant C_{dt} is irrelevant to h_E.

Theorem 3.1. *$\mathcal{A}_h(\cdot\,,\cdot)$, $\mathcal{B}_h(\cdot\,,\cdot)$ and $\mathcal{C}_h(\cdot\,,\cdot)$ are coercive and bounded. If penalty parameter γ satisfies that*

$$\gamma > \gamma_0 = \frac{2(b^\star C_{\text{dt}})^2}{b_\star \min\left\{1, (C_{\text{eq}}^{\text{u}})^{-1} (C_{\text{in}}^{-1})^2\right\}},$$

and h ($h \le 1$) is sufficiently small such that

$$h\left(b^\star C_\Omega C_{\text{dt}}^2 + \gamma C_\Omega C_{\text{dt}}\right) < \frac{1}{2} b_\star \min\left\{1, (C_{\text{eq}}^{\text{u}})^{-1} (C_{\text{in}}^{-1})^2\right\},$$
$$\gamma h C_\Omega C_{\text{dt}} < \gamma - \gamma_0,$$

then $\mathcal{N}_h(\cdot\,,\cdot)$ is coercive and bounded as well.

Proof. The analysis for $\mathcal{A}_h(\cdot\,,\cdot)$ and $\mathcal{C}_h(\cdot\,,\cdot)$ is similar, so we only show the analysis process for $\mathcal{A}_h(\cdot\,,\cdot)$. By (2.1), (3.5) and (3.14), we have for any $v_h, w_h \in V(E)$,

$$\begin{aligned} \left|\mathcal{A}_h^E(v_h, w_h)\right| &\le a^\star \left(\left\|\Pi_0^E v_h\right\|_E \left\|\Pi_0^E w_h\right\|_E + \left(C_{\text{eq}}^{\text{d}}\right)^{-1} \left\|v_h - \Pi_0^E v_h\right\|_E \left\|w_h - \Pi_0^E w_h\right\|_E \right) \\ &\le a^\star \max\left\{1, \left(C_{\text{eq}}^{\text{d}}\right)^{-1}\right\} \|v_h\|_E \|w_h\|_E, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \mathcal{A}_h^E(v_h, v_h) &\ge a_\star \left(\left\|\Pi_0^E v_h\right\|_E^2 + \left(C_{\text{eq}}^{\text{u}}\right)^{-1} \left\|v_h - \Pi_0^E v_h\right\|_E^2 \right) \\ &\ge a_\star \min\left\{1, \left(C_{\text{eq}}^{\text{u}}\right)^{-1}\right\} \|v_h\|_E^2. \end{aligned} \tag{3.17}$$

Applying the Cauchy-Schwarz inequality, (3.16) and (3.17) yields the following continuity and coercivity of $\mathcal{A}_h(\cdot\,,\cdot)$:

$$|\mathcal{A}_h(v_h,w_h)| \le a^\star \max\left\{1, \left(C_{\rm eq}^{\rm d}\right)^{-1}\right\} \|v_h\|_{\Omega_h}\|w_h\|_{\Omega_h}, \quad \forall\, v_h, w_h \in V_{\mathcal{S}}, \tag{3.18}$$

$$\mathcal{A}_h(v_h,v_h) \ge a_\star \min\left\{1, \left(C_{\rm eq}^{\rm u}\right)^{-1}\right\} \|v_h\|_{\Omega_h}^2, \quad \forall\, v_h \in V_{\mathcal{S}}. \tag{3.19}$$

Next, we analyze $\mathcal{B}_h(\cdot\,,\cdot)$. For any $v_h \in V(E) \subset H^1(E)$, we have the following approximation results [10]:

$$\left\|v_h - \Pi_0^E v_h\right\|_E \le C_\nabla h_E |v_h|_{1,E}. \tag{3.20}$$

From (2.1), (3.6), (3.14) and (3.20), we find

$$\begin{aligned}\left|\mathcal{B}_h^E(v_h,w_h)\right| &\le b^\star \left(\left\|\Pi_1^E\nabla v_h\right\|_E\left\|\Pi_1^E\nabla w_h\right\|_E + \left(C_{\rm eq}^{\rm d}\right)^{-1}h_E^{-2}\left\|v_h-\Pi_0^E v_h\right\|_E\left\|w_h-\Pi_0^E w_h\right\|_E\right)\\ &\le b^\star \max\left\{1, \left(C_{\rm eq}^{\rm d}\right)^{-1}C_\nabla^2\right\}|v_h|_{1,E}|w_h|_{1,E}.\end{aligned} \tag{3.21}$$

The following property of Π_1^E follows from its definition in (3.4):

$$\left\|\nabla v_h - \Pi_1^E\nabla v_h\right\|_E \le \left\|\nabla v_h - \nabla\Pi_0^E v_h\right\|_E. \tag{3.22}$$

Employing (2.1), (3.13), (3.14) and (3.22), we have

$$\begin{aligned}\mathcal{B}_h^E(v_h,v_h) &\ge b_\star\left(\left\|\Pi_1^E\nabla v_h\right\|_E^2 + \left(C_{\rm eq}^{\rm u}\right)^{-1}h_E^{-2}\left\|v_h-\Pi_0^E v_h\right\|_E^2\right)\\ &\ge b_\star\left(\left\|\Pi_1^E\nabla v_h\right\|_E^2 + \left(C_{\rm eq}^{\rm u}\right)^{-1}\left(C_{\rm in}^{-1}\right)^2\left\|\nabla v_h-\nabla\Pi_0^E v_h\right\|_E^2\right)\\ &\ge b_\star\left(\left\|\Pi_1^E\nabla v_h\right\|_E^2 + \left(C_{\rm eq}^{\rm u}\right)^{-1}\left(C_{\rm in}^{-1}\right)^2\left\|\nabla v_h-\Pi_1^E\nabla v_h\right\|_E^2\right)\\ &\ge b_\star \min\left\{1, \left(C_{\rm eq}^{\rm u}\right)^{-1}\left(C_{\rm in}^{-1}\right)^2\right\}|v_h|_{1,E}^2.\end{aligned} \tag{3.23}$$

By (3.21), (3.23) and Cauchy-Schwarz inequality, the continuity and coercivity of $\mathcal{B}_h(\cdot\,,\cdot)$ can be derived

$$\left|\mathcal{B}_h(v_h,w_h)\right| \le b^\star \max\left\{1, \left(C_{\rm eq}^{\rm d}\right)^{-1}C_\nabla^2\right\}|v_h|_{1,\Omega_h}|w_h|_{1,\Omega_h}, \quad \forall\, v_h, w_h \in V_{\mathcal{S}}, \tag{3.24}$$

$$\mathcal{B}_h(v_h,v_h) \ge b_\star \min\left\{1, \left(C_{\rm eq}^{\rm u}\right)^{-1}\left(C_{\rm in}^{-1}\right)^2\right\}|v_h|_{1,\Omega_h}^2, \quad \forall\, v_h \in V_{\mathcal{S}}. \tag{3.25}$$

At last, we give the analysis of $\mathcal{N}_h(\cdot\,,\cdot)$. Before proceeding further, we derive some estimates for the boundary terms in $\mathcal{N}_h(\cdot\,,\cdot)$. Using (2.1), (3.6) and (3.15), we have

$$\begin{aligned}\left|\langle b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}, w_h\rangle_{\partial\Omega_h}\right| &\le b^\star\sum_{e\in\mathcal{E}_h^b} h_e^{\frac12}\|\Pi_1\nabla v_h\|_e h_e^{-\frac12}\|w_h\|_e\\ &\le b^\star\left(\sum_{e\in\mathcal{E}_h^b} h_e\|\Pi_1\nabla v_h\|_e^2\right)^{\frac12}|w_h|_{\partial\Omega_h}\\ &\le b^\star C_{\rm dt}|v_h|_{1,\Omega_h}|w_h|_{\partial\Omega_h}.\end{aligned} \tag{3.26}$$

Similar to (3.26), noticing (3.1), we find

$$\begin{aligned}&\left|\langle \rho(\Pi_1\nabla v_h)\cdot\boldsymbol{n}, b(\Pi_1\nabla w_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\right|\\ &\le b^\star h^2 h^{-1}C_\Omega\left(\sum_{e\in\mathcal{E}_h^b} h_e\|\Pi_1\nabla v_h\|_e^2\right)^{\frac12}\left(\sum_{e\in\mathcal{E}_h^b} h_e\|\Pi_1\nabla w_h\|_e^2\right)^{\frac12}\\ &\le b^\star h C_\Omega C_{\rm dt}^2|v_h|_{1,\Omega_h}|w_h|_{1,\Omega_h},\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
&\gamma\Big|\big\langle\rho(\Pi_1\nabla v_h)\cdot\boldsymbol{n},\bar{h}^{-1}w_h\big\rangle_{\partial\Omega_h}\Big|\\
&\le \gamma h^2h^{-1}C_\Omega\left(\sum_{e\in\mathcal{E}_h^b}h_e\|\Pi_1\nabla v_h\|_e^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_h^b}h_e^{-1}\|w_h\|_e^2\right)^{\frac{1}{2}}\\
&\le \gamma hC_\Omega C_{\rm dt}|v_h|_{1,\Omega_h}|w_h|_{\partial\Omega_h}. \qquad (3.28)
\end{aligned}
$$

Based on (3.24) and (3.26)-(3.28), and noticing $h\le 1$, we obtain the following continuity of $\mathcal{N}_h(\cdot\,,\cdot)$ for any $v_h,w_h\in V_{\mathcal{S}}$:

$$
|\mathcal{N}_h(v_h,w_h)|\le\max\left\{b^\star,b^\star\big(C_{\rm eq}^{\rm d}\big)^{-1}C_\nabla^2,b^\star C_{\rm dt},b^\star C_\Omega C_{\rm dt}^2,\gamma C_\Omega C_{\rm dt},\gamma\right\}\|v_h\|_{\mathcal{N}}\|w_h\|_{\mathcal{N}}. \qquad (3.29)
$$

From (3.25) and (3.26)-(3.28), we have

$$
\begin{aligned}
\mathcal{N}_h(v_h,v_h)\ge\ & b_\star\min\left\{1,\big(C_{\rm eq}^{\rm u}\big)^{-1}\big(C_{\rm in}^{-1}\big)^2\right\}|v_h|_{1,\Omega_h}^2-2b^\star C_{\rm dt}|v_h|_{1,\Omega_h}|v_h|_{\partial\Omega_h}+\gamma|v_h|_{\partial\Omega_h}^2\\
&-b^\star hC_\Omega C_{\rm dt}^2|v_h|_{1,\Omega_h}^2-\gamma hC_\Omega C_{\rm dt}|v_h|_{1,\Omega_h}|v_h|_{\partial\Omega_h},
\end{aligned}
$$

then using the modified Young's inequality yields

$$
\begin{aligned}
\mathcal{N}_h(v_h,v_h)\ge&\Big(b_\star\min\left\{1,\big(C_{\rm eq}^{\rm u}\big)^{-1}\big(C_{\rm in}^{-1}\big)^2\right\}-b^\star C_{\rm dt}\varepsilon-h\left(b^\star C_\Omega C_{\rm dt}^2+\gamma C_\Omega C_{\rm dt}\right)\Big)|v_h|_{1,\Omega_h}^2\\
&+\left(\gamma-b^\star C_{\rm dt}\varepsilon^{-1}-\gamma hC_\Omega C_{\rm dt}\right)|v_h|_{\partial\Omega_h}^2.
\end{aligned}
$$

Choose ε as

$$
\varepsilon=\frac{b_\star\min\left\{1,(C_{\rm eq}^{\rm u})^{-1}(C_{\rm in}^{-1})^2\right\}}{2b^\star C_{\rm dt}},
$$

and using the assumption about γ and h and the setting of γ_0, we finally get the coercivity of $\mathcal{N}_h(\cdot\,,\cdot)$ as follows:

$$
\mathcal{N}_h(v_h,v_h)\ge C_{\mathcal{N}}\|v_h\|_{\mathcal{N}}^2,\quad\forall\,v_h\in V_{\mathcal{S}}, \qquad (3.30)
$$

where

$$
C_{\mathcal{N}}:=\min\left\{\frac{1}{2}b_\star\min\left\{1,\big(C_{\rm eq}^{\rm u}\big)^{-1}\big(C_{\rm in}^{-1}\big)^2\right\}-h\left(b^\star C_\Omega C_{\rm dt}^2+\gamma C_\Omega C_{\rm dt}\right),\gamma-\gamma_0-\gamma hC_\Omega C_{\rm dt}\right\}.
$$

The proof is complete. □

3.3. Semi-discrete scheme

For the semi-discrete scheme, we define it by finding $u_h\in L^2[0,T;V_{\mathcal{S}}]$ with $u_{h,t}\in L^2[0,T;V_{\mathcal{S}}]$ satisfying that for any $v_h\in V_{\mathcal{S}}$,

$$
\begin{cases}
\mathcal{A}_h(u_{h,t},v_h)+\mathcal{N}_h(u_{h,t},v_h)+\mathcal{N}_h(u_h,v_h)+\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h),v_h\big)=\mathcal{L}_{\mathcal{N}}(t;v_h),\\
u_h(0)=\mathcal{I}_{\mathcal{S}}u_0,
\end{cases} \qquad (3.31)
$$

where $\mathcal{I}_{\mathcal{S}}u_0$ represents the interpolation of u_0 which is defined in (3.10) and the linear form $\mathcal{L}_{\mathcal{N}}(t;v_h)$ is defined as

$$
\begin{aligned}
\mathcal{L}_{\mathcal{N}}(t;v_h):=&\sum_{E\in\mathcal{T}_h}\int_E\big(\Pi_0^Ef(t)\big)v_h{\rm d}\boldsymbol{x}-\langle\hat{g}_t(t)+\hat{g}(t),b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\
&+\gamma\big\langle\hat{g}_t(t)+\hat{g}(t),\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h},
\end{aligned}
$$

where $\hat{g}(t)=g(\boldsymbol{x}+\rho(\boldsymbol{x})\boldsymbol{n},t)$ with $\hat{g}_t(t)=g_t(\boldsymbol{x}+\rho(\boldsymbol{x})\boldsymbol{n},t)$.

Express a function $\varphi_h \in V_{\mathcal{S}}$ as

$$\varphi_h = \sum_{i=1}^{N_{\mathcal{S}}} \mathtt{dof}_i(\varphi_h)\phi_i,$$

in which ϕ_i represents the i-th global basis function. Then, according to [18], the nonlinear term $\mathcal{C}_h(\mathcal{I}_{\mathcal{S}}c(u_h), v_h)$ can be written as

$$\begin{aligned}\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h), v_h\big) &= \mathcal{C}_h\left(\sum_{i=1}^{N_{\mathcal{S}}} \mathtt{dof}_i\big(\mathcal{I}_{\mathcal{S}}c(u_h)\big)\phi_i, v_h\right)\\ &= \sum_{i=1}^{N_{\mathcal{S}}} \mathtt{dof}_i\big(c(u_h)\big)\mathcal{C}_h(\phi_i, v_h)\\ &= \sum_{i=1}^{N_{\mathcal{S}}} c\big(\mathtt{dof}_i(u_h)\big)\mathcal{C}_h(\phi_i, v_h).\end{aligned}$$

If we further introduce matrix and vector notations as follows:

$$\begin{aligned}
&\boldsymbol{A} = [A_{i,j}] = \mathcal{A}_h(\phi_i, \phi_j), && i,j = 1,2,\ldots,N_{\mathcal{S}}, &&\\
&\boldsymbol{N} = [N_{i,j}] = \mathcal{N}_h(\phi_i, \phi_j), && i,j = 1,2,\ldots,N_{\mathcal{S}}, &&\\
&\boldsymbol{C} = [C_{i,j}] = \mathcal{C}_h(\phi_i, \phi_j), && i,j = 1,2,\ldots,N_{\mathcal{S}}, &&\\
&\boldsymbol{u} = [u_1, u_2, \cdots, u_{N_{\mathcal{S}}}]^{\mathrm{T}}, && u_i = \mathtt{dof}_i(u_h), && i = 1,2,\ldots,N_{\mathcal{S}},\\
&\boldsymbol{u_c} = [(u_c)_1, (u_c)_2, \cdots, (u_c)_{N_{\mathcal{S}}}]^{\mathrm{T}}, && (u_c)_i = c\big(\mathtt{dof}_i(u_h)\big), && i = 1,2,\ldots,N_{\mathcal{S}},\\
&\boldsymbol{L}(t) = [L_1, L_2, \cdots, L_{N_{\mathcal{S}}}]^{\mathrm{T}}, && L_i = \mathcal{L}_{\mathcal{N}}(t;\phi_i), && i = 1,2,\ldots,N_{\mathcal{S}},
\end{aligned}$$

then (3.31) is equivalent to the following ordinary differential equation system:

$$(\boldsymbol{A}+\boldsymbol{N})\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{N}\boldsymbol{u} + \boldsymbol{C}\boldsymbol{u_c} = \boldsymbol{L}(t).$$

From the continuity and coercivity of $\mathcal{A}_h(\cdot\,,\cdot)$, $\mathcal{C}_h(\cdot\,,\cdot)$ and $\mathcal{N}_h(\cdot\,,\cdot)$ given in Theorem 3.1 and the Lipschitz continuity of $c(u)$, we conclude that for (3.31), the solution exists and is unique [27,28].

3.4. Fully discrete scheme

Divide the interval $[0,T]$ into N_T equidistant subintervals with time step size $\Delta t = T/N_T$. The grid-points t^n $(n = 0,1,\ldots,N_T)$ are set to be $t^n = n\Delta t$. Introduce a linear form $\mathcal{L}_{\mathcal{N}}^n(v_h)$ in $V_{\mathcal{S}}$ with the form

$$\begin{aligned}\mathcal{L}_{\mathcal{N}}^n(v_h) \coloneqq{}& \frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E \left(\Pi_0^E f(t^n) + \Pi_0^E f(t^{n-1})\right) v_h \mathrm{d}\boldsymbol{x}\\ &- \frac{1}{\Delta t}\big\langle \hat{g}(t^n) - \hat{g}(t^{n-1}), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}\\ &- \frac{1}{2}\big\langle \hat{g}(t^n) + \hat{g}(t^{n-1}), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h},\end{aligned}$$

then the fully discrete scheme based on the Crank-Nicolson discretization is defined by finding $\{u_h^n\}_{n=1}^{N_T} \subset V_{\mathcal{S}}$ such that for any $v_h \in V_{\mathcal{S}}$,

$$\mathcal{A}_h\left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h\right) + \mathcal{N}_h\left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h\right) + \mathcal{N}_h\left(\frac{u_h^n + u_h^{n-1}}{2}, v_h\right)$$

$$+\mathcal{C}_h\left(\frac{\mathcal{I}_{\mathcal{S}}c(u_h^n)+\mathcal{I}_{\mathcal{S}}c(u_h^{n-1})}{2},v_h\right)=\mathcal{L}_{\mathcal{N}}^n(v_h). \tag{3.32}$$

The initial value $u_h^0=\mathcal{I}_{\mathcal{S}}u_0$. It is obvious that when u_h^{n-1} is obtained, we need to solve the following nonlinear equation to get u_h^n:

$$\mathcal{A}_h\big(u_h^n,v_h\big)+\mathcal{N}_h\big(u_h^n,v_h\big)+\frac{\Delta t}{2}\mathcal{N}_h\big(u_h^n,v_h\big)+\frac{\Delta t}{2}\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c\big(u_h^n\big),v_h\big)=\mathcal{L}_r^n(v_h), \tag{3.33}$$

where

$$\begin{aligned}\mathcal{L}_r^n(v_h)=\mathcal{L}_{\mathcal{N}}^n(v_h)&+\mathcal{A}_h\big(u_h^{n-1},v_h\big)+\mathcal{N}_h\big(u_h^{n-1},v_h\big)\\&-\frac{\Delta t}{2}\mathcal{N}_h\big(u_h^{n-1},v_h\big)-\frac{\Delta t}{2}\mathcal{C}_h\left(\mathcal{I}_{\mathcal{S}}c\big(u_h^{n-1}\big),v_h\right).\end{aligned}$$

Introduce vector notations as follows:

$$\begin{aligned}&\boldsymbol{U}^n=\big[U_1^n,U_2^n,\cdots,U_{N_{\mathcal{S}}}^n\big]^{\mathrm{T}}, && U_i^n=\mathtt{dof}_i\big(u_h^n\big),\quad n=0,1,\ldots,N_T,\\&\boldsymbol{U}_{\boldsymbol{c}}^n=\big[\big(U_c^n\big)_1,\big(U_c^n\big)_2,\cdots,\big(U_c^n\big)_{N_{\mathcal{S}}}\big]^{\mathrm{T}}, && \big(U_c^n\big)_i=c\big(\mathtt{dof}_i\big(u_h^n\big)\big),\\&\boldsymbol{L}^n=\big[L_1^n,L_2^n,\cdots,L_{N_{\mathcal{S}}}^n\big]^{\mathrm{T}}, && L_i^n=\mathcal{L}_{\mathcal{N}}^n(\phi_i),\end{aligned}$$

then (3.33) can be rewritten as

$$\begin{aligned}&\left(\boldsymbol{A}+\boldsymbol{N}+\frac{\Delta t}{2}\boldsymbol{N}\right)\boldsymbol{U}^n+\frac{\Delta t}{2}\boldsymbol{C}\boldsymbol{U}_{\boldsymbol{c}}^n\\&=\boldsymbol{L}^n+\left(\boldsymbol{A}+\boldsymbol{N}-\frac{\Delta t}{2}\boldsymbol{N}\right)\boldsymbol{U}^{n-1}-\frac{\Delta t}{2}\boldsymbol{C}\boldsymbol{U}_{\boldsymbol{c}}^{n-1}.\end{aligned} \tag{3.34}$$

It is clear that when we solve (3.34) by the Newton's iteration, it is easy to construct the Jacobian $\boldsymbol{J}$, and it takes the following form:

$$\boldsymbol{J}=\boldsymbol{A}+\boldsymbol{N}+\frac{\Delta t}{2}\boldsymbol{N}+\frac{\Delta t}{2}\widetilde{\boldsymbol{C}},$$

where

$$\widetilde{\boldsymbol{C}}=\big[\widetilde{C}_{i,j}\big]=C_{i,j}c'\big(\mathtt{dof}_j\big(u_h^n\big)\big),\quad i,j=1,2,\ldots,N_{\mathcal{S}}.$$

Compared with the idea in [2], the strategy of using the interpolation operator to approximate nonlinear terms in [18] can simplify the implementation.

4. Error Analysis

In this section, an energy projection operator $\mathtt{P}$ is first constructed and then error analysis for the discrete schemes (3.31) and (3.32) is derived based on the approximation properties of $\mathtt{P}$. Starting from here, for a positive constant C that does not depend on h, the inequality $\mathfrak{a}\leq C\mathfrak{b}$ is simplified to $\mathfrak{a}\lesssim\mathfrak{b}$. In addition, to carry out the error analysis, we assume that coefficients a and b, source function f, boundary value g, initial value u_0 and the nonlinear term c are regular enough such that the weak solution u of (1.1) can satisfy the regularity that we need.

4.1. The energy projection

The energy projection $\mathtt{P}: H^{3/2+\varepsilon}(\Omega_h) \to V_{\mathcal{S}}$ with $\varepsilon > 0$, is defined by finding $\mathtt{P}w \in V_{\mathcal{S}}$ such that

$$\begin{aligned}\mathcal{N}_h(\mathtt{P}w, v_h) = {}& (b\nabla w, \nabla v_h)_{\Omega_h} - \langle b\nabla w \cdot \boldsymbol{n}, v_h\rangle_{\partial\Omega_h} \\ & - \langle w + \rho\nabla w \cdot \boldsymbol{n}, b(\Pi_1 \nabla v_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + \gamma\big\langle w + \rho\nabla w\cdot\boldsymbol{n}, \bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}. \end{aligned} \tag{4.1}$$

The well-posedness of P can be obtained from (3.26)-(3.28) and the continuity and coercivity of $\mathcal{N}_h(\cdot\,,\cdot)$ in Theorem 3.1. To analyze the approximation properties of P, we recall the following trace inequalities.

Lemma 4.1 ([10]). *If a function $v \in H^1(E)$ for any $E \in \mathcal{T}_h$, then we have*

$$\|v\|_{\partial E} \lesssim h_E^{-\frac{1}{2}}\|v\|_E + h_E^{\frac{1}{2}}|v|_{1,E}. \tag{4.2}$$

For a function $\widetilde{v} \in H^2(E)$, from the Young's inequality, we similarly have

$$\|\nabla\widetilde{v}\|_{\partial E} \lesssim h_E^{-\frac{1}{2}}|\widetilde{v}|_{1,E} + h_E^{\frac{1}{2}}|\widetilde{v}|_{2,E}. \tag{4.3}$$

Lemma 4.2. *Assume that the weak solution u of (1.1) satisfies $u(\cdot,t) \in H^3(\Omega)$, then we have*

$$\|u - \mathtt{P}u\|_{\mathcal{N}} \lesssim h^2\|u\|_{3,\Omega}. \tag{4.4}$$

Proof. Set $\sigma_h = \mathtt{P}u - \mathcal{I}_{\mathcal{S}}u$, then by (3.30) and (4.1), we have

$$\|\sigma_h\|_{\mathcal{N}}^2 \lesssim \mathcal{N}_h(\mathtt{P}u, \sigma_h) - \mathcal{N}_h(\mathcal{I}_{\mathcal{S}}u, \sigma_h) \lesssim Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6,$$

where

$$\begin{aligned} Q_1 &= \big|(b\nabla u, \nabla\sigma_h)_{\Omega_h} - \mathcal{B}_h(\mathcal{I}_{\mathcal{S}}u, \sigma_h)\big|, \\ Q_2 &= \big|\langle b\,(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}u - \nabla u)\cdot\boldsymbol{n}, \sigma_h\rangle_{\partial\Omega_h}\big|, \\ Q_3 &= \big|\langle \mathcal{I}_{\mathcal{S}}u - u, b(\Pi_1\nabla\sigma_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\big|, \\ Q_4 &= \big|\gamma\left\langle u - \mathcal{I}_{\mathcal{S}}u, \bar{h}^{-1}\sigma_h\right\rangle_{\partial\Omega_h}\big|, \\ Q_5 &= \big|\langle \rho(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}u - \nabla u)\cdot\boldsymbol{n}, b(\Pi_1\nabla\sigma_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\big|, \\ Q_6 &= \big|\gamma\left\langle \rho(\nabla u - \Pi_1\nabla\mathcal{I}_{\mathcal{S}}u)\cdot\boldsymbol{n}, \bar{h}^{-1}\sigma_h\right\rangle_{\partial\Omega_h}\big|. \end{aligned}$$

The term Q_1 can be first estimated as

$$\begin{aligned} Q_1 &\lesssim \sum_{E\in\mathcal{T}_h}\big|(b\nabla u, \nabla\sigma_h)_E - \mathcal{B}_h^E(\mathcal{I}_{\mathcal{S}}u, \sigma_h)\big| \\ &= \sum_{E\in\mathcal{T}_h}\big|(b\nabla u, \nabla\sigma_h)_E - \mathcal{B}_h^E\left(\mathcal{I}_{\mathcal{S}}u - \Pi_0^E u, \sigma_h\right) - \left(b\nabla\Pi_0^E u, \Pi_1^E\nabla\sigma_h\right)_E\big| \\ &\lesssim \sum_{E\in\mathcal{T}_h}\big|\left(b\nabla u - \Pi_1^E(b\nabla u), \nabla\sigma_h\right)_E\big| + \big|\mathcal{B}_h^E\left(\mathcal{I}_{\mathcal{S}}u - \Pi_0^E u, \sigma_h\right)\big| \\ &\quad + \sum_{E\in\mathcal{T}_h}\big|\left(b\left(\nabla u - \nabla\Pi_0^E u\right), \Pi_1^E\nabla\sigma_h\right)_E\big|, \end{aligned}$$

then using (3.6)-(3.8), (3.11) and (3.21) yields

$$Q_1 \lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}.$$

By (3.6), (3.15) and (4.3), we have

$$\begin{aligned}
Q_2 &\lesssim \sum_{e\in\mathcal{E}_h^b} \left\|\Pi_1^{E_e}\nabla\mathcal{I}_{\mathcal{S}}u - \nabla u\right\|_e \|\sigma_h\|_e \\
&\lesssim \sum_{e\in\mathcal{E}_h^b} \left(\left\|\Pi_1^{E_e}\nabla\mathcal{I}_{\mathcal{S}}u - \nabla\Pi_0^{E_e}u\right\|_e + \left\|\nabla\Pi_0^{E_e}u - \nabla u\right\|_e\right) \|\sigma_h\|_e \\
&\lesssim \sum_{e\in\mathcal{E}_h^b} \left(\left\|\Pi_1^{E_e}\nabla\mathcal{I}_{\mathcal{S}}u - \nabla\Pi_0^{E_e}u\right\|_{E_e} + \left|\Pi_0^{E_e}u - u\right|_{1,E_e}\right) h_e^{-\frac{1}{2}} \|\sigma_h\|_e \\
&\quad + \sum_{e\in\mathcal{E}_h^b} h_{E_e} \left|\Pi_0^{E_e}u - u\right|_{2,E_e} h_e^{-\frac{1}{2}} \|\sigma_h\|_e,
\end{aligned}$$

where E_e is the unique mesh element satisfying $e \subset \partial E_e$. Utilizing (3.7), (3.8) and (3.11), we arrive at

$$Q_2 \lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}.$$

From (3.15) and (4.3), we see that

$$\begin{aligned}
Q_3 &\lesssim \sum_{e\in\mathcal{E}_h^b} \|\mathcal{I}_{\mathcal{S}}u - u\|_e \left\|\Pi_1^{E_e}\nabla\sigma_h\right\|_e \\
&\lesssim \sum_{e\in\mathcal{E}_h^b} \left(h_{E_e}^{-1}\|\mathcal{I}_{\mathcal{S}}u - u\|_{E_e} + |\mathcal{I}_{\mathcal{S}}u - u|_{1,E_e}\right) \|\Pi_1^{E_e}\nabla\sigma_h\|_{E_e},
\end{aligned}$$

then applying (3.6) and (3.11), we obtain

$$Q_3 \lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}.$$

Similar to Q_3, we can bound Q_4 as

$$\begin{aligned}
Q_4 &\lesssim \sum_{e\in\mathcal{E}_h^b} \left(h_{E_e}^{-1}\|\mathcal{I}_{\mathcal{S}}u - u\|_{E_e} + |\mathcal{I}_{\mathcal{S}}u - u|_{1,E_e}\right) h_e^{-\frac{1}{2}} \|\sigma_h\|_{E_e} \\
&\lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}.
\end{aligned}$$

Noticing that $\rho \lesssim h^2 \leq h$, we can estimate Q_5 as

$$\begin{aligned}
Q_5 &\lesssim \sum_{e\in\mathcal{E}_h^b} h\left\|\Pi_1^{E_e}\nabla\mathcal{I}_{\mathcal{S}}u - \nabla u\right\|_e \left\|\Pi_1^{E_e}\nabla\sigma_h\right\|_e \\
&\lesssim \sum_{e\in\mathcal{E}_h^b} \left(\left\|\Pi_1^{E_e}\nabla u - \nabla u\right\|_{E_e} + |u - \mathcal{I}_{\mathcal{S}}u|_{1,E_e}\right) \left\|\Pi_1^{E_e}\nabla\sigma_h\right\|_{E_e} \\
&\quad + \sum_{e\in\mathcal{E}_h^b} \left(\left|\Pi_0^{E_e}u - u\right|_{1,E_e} + h_{E_e}\left|\Pi_0^{E_e}u - u\right|_{2,E_e}\right) \left\|\Pi_1^{E_e}\nabla\sigma_h\right\|_{E_e},
\end{aligned}$$

then applying (3.6)-(3.8) and (3.11), we get

$$Q_5 \lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}.$$

Similarly, Q_6 can be bounded as

$$\begin{aligned} Q_6 &\lesssim \sum_{e\in\mathcal{E}_h^b} h^{\frac{1}{2}} \left\|\nabla u - \Pi_1^{E_e}\nabla \mathcal{I}_{\mathcal{S}} u\right\|_e h_e^{-\frac{1}{2}} \|\sigma_h\|_e \\ &\lesssim h^2 \|u\|_{3,\Omega} \|\sigma_h\|_{\mathcal{N}}. \end{aligned}$$

On the other hand, from (3.11) and (4.2), it is straightforward to obtain

$$\|u - \mathcal{I}_{\mathcal{S}} u\|_{\mathcal{N}} \lesssim |u - \mathcal{I}_{\mathcal{S}} u|_{1,\Omega_h} + |u - \mathcal{I}_{\mathcal{S}} u|_{\partial\Omega_h} \lesssim h^2 \|u\|_{3,\Omega}.$$

Finally, (4.4) is derived by using the triangular inequality. □

Lemma 4.3. *Suppose that for* (1.1)*, the weak solution u satisfies $u(\cdot,t) \in H^3(\Omega)$, then we have*

$$\|u - \mathtt{P}u\|_{\Omega_h} \lesssim h^3 \|u\|_{3,\Omega}. \tag{4.5}$$

Proof. Consider the following dual problem with the solution $\zeta \in H^2(\Omega_h) \cap H_0^1(\Omega_h)$:

$$\begin{aligned} -\nabla \cdot \big(b(\boldsymbol{x})\nabla\zeta\big) &= u - \mathtt{P}u, \quad && \boldsymbol{x} \in \Omega_h, \\ \zeta &= 0, && \boldsymbol{x} \in \partial\Omega_h. \end{aligned}$$

The following regularity bound follows from the convexity of Ω_h:

$$\|\zeta\|_{2,\Omega_h} \lesssim \|u - \mathtt{P}u\|_{\Omega_h}. \tag{4.6}$$

Let $\mathcal{I}_{\mathcal{S}}\zeta$ represent the interpolation of ζ in $V_{\mathcal{S}}$, then we have $\mathcal{I}_{\mathcal{S}}\zeta|_{\partial\Omega_h} = 0$ by the boundary condition for ζ and the definition of the global degrees of freedom for $V_{\mathcal{S}}$. In light of (3.11) and (4.6), we further have

$$|\zeta - \mathcal{I}_{\mathcal{S}}\zeta|_{1,\Omega_h} \lesssim h \|u - \mathtt{P}u\|_{\Omega_h}. \tag{4.7}$$

By the definition of $\mathtt{P}$ and $\mathcal{N}_h(\cdot\,,\cdot)$ and the fact that $\mathcal{I}_{\mathcal{S}}\zeta|_{\partial\Omega_h} = 0$, we have

$$\begin{aligned} \|u - \mathtt{P}u\|_{\Omega_h}^2 &= \big(u - \mathtt{P}u, -\nabla\cdot(b\nabla\zeta)\big)_{\Omega_h} \\ &= \big(b\nabla(u - \mathtt{P}u), \nabla\zeta\big)_{\Omega_h} - \langle b\nabla\zeta\cdot\boldsymbol{n}, u - \mathtt{P}u\rangle_{\partial\Omega_h} \\ &= \big(b\nabla(u - \mathtt{P}u), \nabla(\zeta - \mathcal{I}_{\mathcal{S}}\zeta)\big)_{\Omega_h} + \big(b\nabla(u - \mathtt{P}u), \nabla\mathcal{I}_{\mathcal{S}}\zeta\big)_{\Omega_h} - \langle b\nabla\zeta\cdot\boldsymbol{n}, u - \mathtt{P}u\rangle_{\partial\Omega_h} \\ &= \big(b\nabla(u - \mathtt{P}u), \nabla(\zeta - \mathcal{I}_{\mathcal{S}}\zeta)\big)_{\Omega_h} - (b\nabla\mathtt{P}u, \nabla\mathcal{I}_{\mathcal{S}}\zeta)_{\Omega_h} - \langle b\nabla\zeta\cdot\boldsymbol{n}, u - \mathtt{P}u\rangle_{\partial\Omega_h} \\ &\quad + \mathcal{N}_h(\mathtt{P}u, \mathcal{I}_{\mathcal{S}}\zeta) + \langle u, b(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}\zeta)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \rho\nabla u\cdot\boldsymbol{n}, b(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}\zeta)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} \\ &\lesssim R_1 + R_2 + R_3 + R_4, \end{aligned}$$

where

$$\begin{aligned} R_1 &= \left|\big(b\nabla(u - \mathtt{P}u), \nabla(\zeta - \mathcal{I}_{\mathcal{S}}\zeta)\big)_{\Omega_h}\right|, \\ R_2 &= \left|\mathcal{B}_h(\mathtt{P}u, \mathcal{I}_{\mathcal{S}}\zeta) - (b\nabla\mathtt{P}u, \nabla\mathcal{I}_{\mathcal{S}}\zeta)_{\Omega_h}\right|, \\ R_3 &= \left|\langle u - \mathtt{P}u, b(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}\zeta - \nabla\zeta)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\right|, \\ R_4 &= \left|\langle \rho(\nabla u - \Pi_1\nabla\mathtt{P}u)\cdot\boldsymbol{n}, b(\Pi_1\nabla\mathcal{I}_{\mathcal{S}}\zeta)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\right|. \end{aligned}$$

By (4.4) and (4.7), for R_1, we have

$$R_1 \lesssim |u - \mathtt{P}u|_{1,\Omega_h} |\zeta - \mathcal{I}_{\mathcal{S}}\zeta|_{1,\Omega_h}$$

$$\lesssim \|u - \mathtt{P}u\|_{\mathcal{N}} |\zeta - \mathcal{I}_{\mathcal{S}}\zeta|_{1,\Omega_h} \lesssim h^3 |u|_{3,\Omega} \|u - \mathtt{P}u\|_{\Omega_h}.$$

In the same way as the inconsistency error analysis established in [12, Theorem 6.3], R_2 can be bounded as

$$R_2 \lesssim h^3 |u|_{3,\Omega} \|u - \mathtt{P}u\|_{\Omega_h}.$$

Based on (3.6), (3.7), (3.15), (4.3), (4.4) and (4.7), we have

$$\begin{aligned}
R_3 &\lesssim \sum_{e \in \mathcal{E}_h^b} h_e^{\frac{1}{2}} \left\| \Pi_1^{E_e} \nabla \mathcal{I}_{\mathcal{S}} \zeta - \nabla \zeta \right\|_e h_e^{-\frac{1}{2}} \|u - \mathtt{P}u\|_e \\
&\lesssim \sum_{e \in \mathcal{E}_h^b} h_e^{\frac{1}{2}} \left(\left\| \Pi_1^{E_e} \nabla \mathcal{I}_{\mathcal{S}} \zeta - \nabla \Pi_0^{E_e} \zeta \right\|_e + \left\| \nabla \Pi_0^{E_e} \zeta - \nabla \zeta \right\|_e \right) h_e^{-\frac{1}{2}} \|u - \mathtt{P}u\|_e \\
&\lesssim \sum_{e \in \mathcal{E}_h^b} \left(\left\| \Pi_1^{E_e} \nabla \zeta - \nabla \zeta \right\|_{E_e} + |\mathcal{I}_{\mathcal{S}} \zeta - \zeta|_{1,E_e} \right) h_e^{-\frac{1}{2}} \|u - \mathtt{P}u\|_e \\
&\quad + \sum_{e \in \mathcal{E}_h^b} \left(\left| \Pi_0^{E_e} \zeta - \zeta \right|_{1,E_e} + h_{E_e} \left| \Pi_0^{E_e} \zeta - \zeta \right|_{2,E_e} \right) h_e^{-\frac{1}{2}} \|u - \mathtt{P}u\|_e \\
&\lesssim h |\zeta|_{2,\Omega_h} \|u - \mathtt{P}u\|_{\mathcal{N}} \lesssim h^3 |u|_{3,\Omega} \|u - \mathtt{P}u\|_{\Omega_h}.
\end{aligned}$$

Similarly, R_4 can be estimated as

$$\begin{aligned}
R_4 &\lesssim \sum_{e \in \mathcal{E}_h^b} h^2 h_e^{-\frac{1}{2}} \left\| \nabla u - \Pi_1^{E_e} \nabla \mathtt{P}u \right\|_e h_e^{\frac{1}{2}} \left\| \Pi_1^{E_e} \nabla \mathcal{I}_{\mathcal{S}} \zeta \right\|_e \\
&\lesssim \sum_{e \in \mathcal{E}_h^b} h^{\frac{3}{2}} \left(\left\| \nabla u - \Pi_1^{E_e} \nabla u \right\|_e + \left\| \Pi_1^{E_e} \nabla u - \Pi_1^{E_e} \nabla \mathtt{P}u \right\|_e \right) \|\nabla \mathcal{I}_{\mathcal{S}} \zeta\|_{E_e} \\
&\lesssim \sum_{e \in \mathcal{E}_h^b} h^{\frac{3}{2}} \left(\left\| \nabla u - \nabla \Pi_0^{E_e} u \right\|_e + \left\| \nabla \Pi_0^{E_e} u - \Pi_1^{E_e} \nabla u \right\|_e \right) \|\nabla \mathcal{I}_{\mathcal{S}} \zeta\|_{E_e} \\
&\quad + \sum_{e \in \mathcal{E}_h^b} h^{\frac{3}{2}} \left\| \Pi_1^{E_e} \nabla u - \Pi_1^{E_e} \nabla \mathtt{P}u \right\|_e \|\nabla \mathcal{I}_{\mathcal{S}} \zeta\|_{E_e} \\
&\lesssim h^3 |u|_{3,\Omega} |\mathcal{I}_{\mathcal{S}} \zeta|_{1,\Omega_h} \lesssim h^3 |u|_{3,\Omega} \big(|\mathcal{I}_{\mathcal{S}} \zeta - \zeta|_{1,\Omega_h} + |\zeta|_{1,\Omega_h} \big) \lesssim h^3 |u|_{3,\Omega} \|u - \mathtt{P}u\|_{\Omega_h}.
\end{aligned}$$

Using the above estimates for R_1-R_4, we get (4.5).

4.2. Error analysis for the semi-discrete scheme

Some error estimates for terms involving the boundary condition are first derived in this subsection, then the error analysis is presented for (3.31) on the basis of approximation properties (4.4) and (4.5) of the energy projection $\mathtt{P}$.

Lemma 4.4. *Assume that the weak solution u of (1.1) satisfies $u(\cdot,t) \in W^{2,\infty}(\Omega)$, and the time derivative $u_t(\cdot,t)$ is assumed to belong to $W^{2,\infty}(\Omega)$ as well, then we have for any $v_h \in V_{\mathcal{S}}$,*

$$\begin{aligned}
&\big\langle \hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho \nabla (u_t + u) \cdot \boldsymbol{n} \big), b(\Pi_1 \nabla v_h) \cdot \boldsymbol{n} \big\rangle_{\partial \Omega_h} \\
&\lesssim h^3 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega} \big) \|v_h\|_{\Omega_h}, \qquad (4.8) \\
&-\gamma \big\langle \hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho \nabla (u_t + u) \cdot \boldsymbol{n} \big), \bar{h}^{-1} v_h \big\rangle_{\partial \Omega_h} \\
&\lesssim h^3 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega} \big) \|v_h\|_{\Omega_h}. \qquad (4.9)
\end{aligned}$$

Proof. According to Taylor expansion [8] and (3.1), for $\boldsymbol{x} \in \partial\Omega_h$, it holds

$$
\begin{aligned}
|\hat{g}(t) - (u + \rho\nabla u \cdot \boldsymbol{n})| &= \big|u\big(\boldsymbol{x} + \rho(\boldsymbol{x})\boldsymbol{n}, t\big) - (u + \rho\nabla u \cdot \boldsymbol{n})\big| \\
&\lesssim \rho^2 |u|_{2,\infty,\Omega} \lesssim h^4 |u|_{2,\infty,\Omega}, \qquad (4.10)\\
|\hat{g}_t(t) - (u_t + \rho\nabla u_t \cdot \boldsymbol{n})| &= \big|u_t\big(\boldsymbol{x} + \rho(\boldsymbol{x})\boldsymbol{n}, t\big) - (u_t + \rho\nabla u_t \cdot \boldsymbol{n})\big| \\
&\lesssim \rho^2 |u_t|_{2,\infty,\Omega} \lesssim h^4 |u_t|_{2,\infty,\Omega}. \qquad (4.11)
\end{aligned}
$$

Then, by (3.6), (3.13), (4.10) and (4.11), we have

$$
\begin{aligned}
&\big\langle \hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho\nabla(u_t + u)\cdot \boldsymbol{n}\big), b(\Pi_1 \nabla v_h)\cdot \boldsymbol{n}\big\rangle_{\partial\Omega_h} \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} \big\|\hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho\nabla(u_t + u)\cdot \boldsymbol{n}\big)\big\|_e \big\|\Pi_1^{E_e}\nabla v_h\big\|_e \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} h^4 h_e^{\frac{1}{2}} \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big)\big\|\Pi_1^{E_e}\nabla v_h\big\|_e \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} h^4 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big)\|\nabla v_h\|_{E_e} \\
\lesssim &\, h^3 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big)\|v_h\|_{\Omega_h}.
\end{aligned}
$$

Thus, (4.8) is proven. Employing (4.2) further, we can obtain (4.9) in the following way:

$$
\begin{aligned}
&-\gamma\big\langle \hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho\nabla(u_t + u)\cdot \boldsymbol{n}\big), \bar{h}^{-1} v_h\big\rangle_{\partial\Omega_h} \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} \big\|\hat{g}_t(t) + \hat{g}(t) - \big(u_t + u + \rho\nabla(u_t + u)\cdot \boldsymbol{n}\big)\big\|_e h_e^{-1}\|v_h\|_e \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} h^4 h_e^{\frac{1}{2}} \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big) h_e^{-1}\|v_h\|_e \\
\lesssim &\sum_{e\in\mathcal{E}_h^b} h^4 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big)\big(h_{E_e}^{-1}\|v_h\|_{E_e} + |v_h|_{1,E_e}\big) \\
\lesssim &\, h^3 \big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big)\|v_h\|_{\Omega_h}.
\end{aligned}
$$

The proof is complete. □

Theorem 4.1. *Let u be the solution of* (1.1) *and u_h be the solution of* (3.31). *Suppose that $u \in L^2[0,T;H^3(\Omega)\cap W^{2,\infty}(\Omega)]$, $u_t \in L^2[0,T;H^3(\Omega)\cap W^{2,\infty}(\Omega)]$, $c(u) \in L^2[0,T;H^3(\Omega)]$, $f \in L^2[0,T;H^3(\Omega)]$ and $u_0 \in H^3(\Omega)$, then for any t with $0 < t < T$, the following error estimates hold:*

$$
\begin{aligned}
\|u - u_h\|_{\Omega_h} \lesssim h^3 \big(&\|u_0\|_{3,\Omega} + \|u_t\|_{L^2[0,t;H^3(\Omega)]} + \|f\|_{L^2[0,t;H^3(\Omega)]} + \|u\|_{L^2[0,t;H^3(\Omega)]} \\
&+ \|c(u)\|_{L^2[0,t;H^3(\Omega)]} + \|u\|_{L^2[0,t;W^{2,\infty}(\Omega)]} \\
&+ \|u_t\|_{L^2[0,t;W^{2,\infty}(\Omega)]} + \|u_t\|_{L^1[0,t;H^3(\Omega)]}\big), \qquad (4.12)\\
\|u - u_h\|_{\mathcal{N}} \lesssim h^3 \big(&\|u_0\|_{3,\Omega} + \|u_t\|_{L^2[0,t;H^3(\Omega)]} + \|f\|_{L^2[0,t;H^3(\Omega)]} + \|u\|_{L^2[0,t;H^3(\Omega)]} \\
&+ \|c(u)\|_{L^2[0,t;H^3(\Omega)]} + \|u\|_{L^2[0,t;W^{2,\infty}(\Omega)]} + \|u_t\|_{L^2[0,t;W^{2,\infty}(\Omega)]}\big) \\
+ h^2 &\left(\|u_0\|_{3,\Omega} + \|u_t\|_{L^1[0,t;H^3(\Omega)]}\right). \qquad (4.13)
\end{aligned}
$$

Proof. Utilizing the energy projection $\mathtt{P}$, the error $u(t)-u_h(t)$ can be split as

$$u(t)-u_h(t)=u(t)-\mathtt{P}u(t)+\mathtt{P}u(t)-u_h(t).$$

Thanks to Lemmas 4.2 and 4.3, it follows that

$$\|u-\mathtt{P}u\|_{\mathcal{N}}\lesssim h^2\|u\|_{3,\Omega}\lesssim h^2\left(\|u_0\|_{3,\Omega}+\|u_t\|_{L^1[0,t;H^3(\Omega)]}\right), \tag{4.14}$$

$$\|u-\mathtt{P}u\|_{\Omega_h}\lesssim h^3\|u\|_{3,\Omega}\lesssim h^3\left(\|u_0\|_{3,\Omega}+\|u_t\|_{L^1[0,t;H^3(\Omega)]}\right). \tag{4.15}$$

For the sake of convenience, we set $\theta(t)=\mathtt{P}u(t)-u_h(t)$. Based on the semi-discrete scheme (3.31) and the definition (4.1) of $\mathtt{P}$, a simple calculation gives

$$\begin{aligned}
&\mathcal{A}_h(\theta_t,v_h)+\mathcal{N}_h(\theta_t,v_h)+\mathcal{N}_h(\theta,v_h)\\
&=\mathcal{A}_h(\mathtt{P}u_t,v_h)+\mathcal{N}_h(\mathtt{P}u_t,v_h)+\mathcal{N}_h(\mathtt{P}u,v_h)+\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h),v_h\big)-\mathcal{L}_{\mathcal{N}}(t;v_h)\\
&=\mathcal{A}_h(\mathtt{P}u_t,v_h)+\big(b\nabla(u_t+u),\nabla v_h\big)_{\Omega_h}-\langle b\nabla(u_t+u)\cdot\boldsymbol{n},v_h\rangle_{\partial\Omega_h}+\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h),v_h\big)\\
&\quad-\sum_{E\in\mathcal{T}_h}\int_E\big(\Pi_0^Ef(t)\big)v_h\mathrm{d}\boldsymbol{x}+\big\langle\hat{g}_t(t)+\hat{g}(t),b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}-\gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}\\
&\quad-\big\langle u_t+u+\rho\nabla(u_t+u)\cdot\boldsymbol{n},b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}-\gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}.
\end{aligned}$$

For the model problem (1.1), it is easy to verify that its solution u satisfies that for any $v_h\in V_{\mathcal{S}}$,

$$(au_t,v_h)_{\Omega_h}+\big(b\nabla(u_t+u),\nabla v_h\big)_{\Omega_h}-\langle b\nabla(u_t+u)\cdot\boldsymbol{n},v_h\rangle_{\partial\Omega_h}+\big(c(u),v_h\big)_{\Omega_h}=(f,v_h)_{\Omega_h},$$

then we have

$$\mathcal{A}_h(\theta_t,v_h)+\mathcal{N}_h(\theta_t,v_h)+\mathcal{N}_h(\theta,v_h)=\mathcal{X}_1+\mathcal{X}_2+\mathcal{X}_3+\mathcal{X}_4+\mathcal{X}_5,$$

where

$$\begin{aligned}
\mathcal{X}_1&=\mathcal{A}_h(\mathtt{P}u_t,v_h)-(au_t,v_h)_{\Omega_h},\\
\mathcal{X}_2&=\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c\big(u_h(t)\big),v_h\big)-\big(c(u),v_h\big)_{\Omega_h},\\
\mathcal{X}_3&=(f,v_h)_{\Omega_h}-\sum_{E\in\mathcal{T}_h}\int_E\big(\Pi_0^Ef(t)\big)v_h\mathrm{d}\boldsymbol{x},\\
\mathcal{X}_4&=-\gamma\big\langle\hat{g}_t(t)+\hat{g}(t)-\big(u_t+u+\rho\nabla(u_t+u)\cdot\boldsymbol{n}\big),\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h},\\
\mathcal{X}_5&=\big\langle\hat{g}_t(t)+\hat{g}(t)-\big(u_t+u+\rho\nabla(u_t+u)\cdot\boldsymbol{n}\big),b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}\big\rangle_{\partial\Omega_h}.
\end{aligned}$$

Using (3.5), (3.8) and (4.5), we obtain

$$\begin{aligned}
\mathcal{X}_1&=\sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E(\mathtt{P}u_t,v_h)-(au_t,v_h)_E\\
&=\sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E\big(\mathtt{P}u_t-\Pi_0^Eu_t,v_h\big)+\big(a\Pi_0^Eu_t,\Pi_0^Ev_h\big)_E-(au_t,v_h)_E\\
&=\sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E\big(\mathtt{P}u_t-\Pi_0^Eu_t,v_h\big)+\big(a\Pi_0^Eu_t-au_t,\Pi_0^Ev_h\big)_E-\big(au_t-\Pi_0^E(au_t),v_h\big)_E\\
&\lesssim h^3\|u_t\|_{3,\Omega}\|v_h\|_{\Omega_h}.
\end{aligned}$$

For $\mathcal{X}_2$, in [18, Theorem 1], it has been estimated as

$$\mathcal{X}_2 \lesssim \big(h^3\|c(u)\|_{3,\Omega} + h^3\|u\|_{3,\Omega} + \|u-u_h\|_{\Omega_h}\big)\|v_h\|_{\Omega_h}.$$

By (3.8), we can simply estimate $\mathcal{X}_3$ as

$$\mathcal{X}_3 = \sum_{E\in\mathcal{T}_h}\int_E \big(f(t)-\Pi_0^E f(t)\big)v_h \mathrm{d}\boldsymbol{x} \lesssim h^3\|f(t)\|_{3,\Omega}\|v_h\|_{\Omega_h}.$$

The error terms $\mathcal{X}_4$ and $\mathcal{X}_5$ arise from the deformation of $\partial\Omega$ to $\partial\Omega_h$ and their estimates are provided in Lemma 4.4. Combining with the above analysis for $\mathcal{X}_1$-$\mathcal{X}_3$, we arrive at

$$\begin{aligned}\mathcal{A}_h(\theta_t, v_h) + \mathcal{N}_h(\theta_t+\theta, v_h) \lesssim \Big(&h^3\big(\|u_t\|_{3,\Omega} + \|f(t)\|_{3,\Omega} + \|u\|_{3,\Omega} + \|c(u)\|_{3,\Omega}\big)\\ &+ h^3\big(|u|_{2,\infty,\Omega} + |u_t|_{2,\infty,\Omega}\big) + \|u-u_h\|_{\Omega_h}\Big)\|v_h\|_{\Omega_h}. \end{aligned} \tag{4.16}$$

Taking the place of v_h with $\theta_t + \theta$ in (4.16) and noticing the coercivity of $\mathcal{A}_h(\cdot\,,\cdot)$ and $\mathcal{N}_h(\cdot\,,\cdot)$, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_h(\theta,\theta) \lesssim{}& h^6\big(\|u_t\|_{3,\Omega}^2 + \|f(t)\|_{3,\Omega}^2 + \|u\|_{3,\Omega}^2 + \|c(u)\|_{3,\Omega}^2 + |u|_{2,\infty,\Omega}^2 + |u_t|_{2,\infty,\Omega}^2\big)\\ &+ \|u-u_h\|_{\Omega_h}^2 + \|\theta\|_{\Omega_h}^2,\end{aligned}$$

then by (4.15), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_h(\theta,\theta) \lesssim h^6\big(&\|u_t\|_{3,\Omega}^2 + \|f(t)\|_{3,\Omega}^2 + \|u\|_{3,\Omega}^2 + \|c(u)\|_{3,\Omega}^2\\ &+ |u|_{2,\infty,\Omega}^2 + |u_t|_{2,\infty,\Omega}^2\big) + \|\theta\|_{\Omega_h}^2.\end{aligned} \tag{4.17}$$

Integrating (4.17) on $(0,t)$ and using the continuous Grönwall's lemma, it holds

$$\begin{aligned}\|\theta\|_{\Omega_h} \lesssim{}& h^3\left(\int_0^t \big(\|u_t(\tau)\|_{3,\Omega}^2 + \|f(\tau)\|_{3,\Omega}^2 + \|u(\tau)\|_{3,\Omega}^2 + \big\|c\big(u(\tau)\big)\big\|_{3,\Omega}^2\big)\mathrm{d}\tau\right)^{\frac12}\\ &+ h^3\left(\int_0^t \big(|u_t(\tau)|_{2,\infty,\Omega}^2 + |u(\tau)|_{2,\infty,\Omega}^2\big)\mathrm{d}\tau\right)^{\frac12} + \|\theta(0)\|_{\Omega_h}.\end{aligned} \tag{4.18}$$

Based on (3.11) and (4.5), we have

$$\|\theta(0)\|_{\Omega_h} = \|\mathtt{P}u_0 - \mathcal{I}_{\mathcal{S}}u_0\|_{\Omega_h} \le \|\mathtt{P}u_0 - u_0\|_{\Omega_h} + \|u_0 - \mathcal{I}_{\mathcal{S}}u_0\|_{\Omega_h} \lesssim h^3\|u_0\|_{3,\Omega}. \tag{4.19}$$

Applying (4.15), (4.18) and (4.19), we obtain (4.12).

Next, we focus on (4.13). Replacing v_h with θ_t in (4.16), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}_h(\theta,\theta) \lesssim{}& h^6\left(\|u_t\|_{3,\Omega}^2 + \|f(t)\|_{3,\Omega}^2 + \|u\|_{3,\Omega}^2 + \|c(u)\|_{3,\Omega}^2 + |u|_{2,\infty,\Omega}^2 + |u_t|_{2,\infty,\Omega}^2\right)\\ &+ h^6\|u_0\|_{3,\Omega}^2 + h^6\int_0^t \big(\|u_t(\tau)\|_{3,\Omega}^2 + \|f(\tau)\|_{3,\Omega}^2 + \|u(\tau)\|_{3,\Omega}^2\big)\mathrm{d}\tau\\ &+ h^6\int_0^t \Big(\big\|c\big(u(\tau)\big)\big\|_{3,\Omega}^2 + |u(\tau)|_{2,\infty,\Omega}^2 + |u_t(\tau)|_{2,\infty,\Omega}^2\Big)\mathrm{d}\tau.\end{aligned}$$

Integrating the above bound form 0 to t, we have

$$\|\theta\|_{\mathcal{N}}^2 \lesssim h^6\|u_0\|_{3,\Omega}^2 + h^6\int_0^t \big(\|u_t(\tau)\|_{3,\Omega}^2 + \|f(\tau)\|_{3,\Omega}^2 + \|u(\tau)\|_{3,\Omega}^2\big)\mathrm{d}\tau + \|u_0 - \mathcal{I}_{\mathcal{S}}u_0\|_{\mathcal{N}}^2$$

$$
\begin{aligned}
&+ h^6 \int_0^t \Big(\big\|c\big(u(\tau)\big)\big\|_{3,\Omega}^2 + |u(\tau)|_{2,\infty,\Omega}^2 + |u_t(\tau)|_{2,\infty,\Omega}^2\Big)\mathrm{d}\tau + \|\mathrm{P}u_0 - u_0\|_{\mathcal{N}}^2 \\
\lesssim & h^6 \int_0^t \big(\|u_t(\tau)\|_{3,\Omega}^2 + \|f(\tau)\|_{3,\Omega}^2 + \|u(\tau)\|_{3,\Omega}^2\big)\mathrm{d}\tau + h^6\|u_0\|_{3,\Omega}^2 + h^4\|u_0\|_{3,\Omega}^2 \\
&+ h^6 \int_0^t \Big(\big\|c\big(u(\tau)\big)\big\|_{3,\Omega}^2 + |u(\tau)|_{2,\infty,\Omega}^2 + |u_t(\tau)|_{2,\infty,\Omega}^2\Big)\mathrm{d}\tau,
\end{aligned}
$$

then, combining with (4.14), we obtain (4.13). □

4.3. Error analysis for the fully discrete scheme

Before carrying out the error analysis for the fully discrete scheme (3.32), we establish some error estimates for terms involving the boundary condition.

Lemma 4.5. *Let u be the weak solution of* (1.1)*. Assume that for $n = 0, 1, \ldots, N_T$, there is a positive constant C_u^∞ such that $\max_{n-1\le i\le n}|u(t^i)|_{2,\infty,\Omega} \le C_u^\infty$ and $u_t \in L^1[t^{n-1}, t^n; W^{2,\infty}(\Omega)]$, then for any $v_h \in V_{\mathcal{S}}$, we have*

$$
\begin{aligned}
&\left\langle \frac{1}{\Delta t}\Big(\hat g(t^n) - \hat g(t^{n-1}) - \big(u(t^n) - u(t^{n-1}) + \rho\nabla\big(u(t^n) - u(t^{n-1})\big)\cdot \boldsymbol{n}\big)\Big), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \frac{\gamma v_h}{\bar h}\right\rangle_{\partial\Omega_h} \\
&\lesssim \frac{h^3}{\Delta t}\|v_h\|_{\Omega_h}\int_{t^{n-1}}^{t^n} |u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau,
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
&\left\langle \frac{1}{2}\Big(\hat g(t^n) + \hat g(t^{n-1}) - \big(u(t^n) + u(t^{n-1}) + \rho\nabla\big(u(t^n) + u(t^{n-1})\big)\cdot \boldsymbol{n}\big)\Big), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \frac{\gamma v_h}{\bar h}\right\rangle_{\partial\Omega_h} \\
&\lesssim h^3 \max_{n-1\le i\le n}|u(t^i)|_{2,\infty,\Omega}\|v_h\|_{\Omega_h}.
\end{aligned}
\tag{4.21}
$$

Proof. According to (3.6), (3.13), (3.15) and (4.11), we have

$$
\begin{aligned}
&\left\langle \frac{1}{\Delta t}\Big(\hat g(t^n) - \hat g(t^{n-1}) - \big(u(t^n) - u(t^{n-1}) + \rho\nabla\big(u(t^n) - u(t^{n-1})\big)\cdot \boldsymbol{n}\big)\Big), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}\right\rangle_{\partial\Omega_h} \\
= & \frac{1}{\Delta t}\left\langle \int_{t^{n-1}}^{t^n} \hat g_t(\tau) - \big(u_t(\tau) + \rho\nabla u_t(\tau)\cdot\boldsymbol{n}\big)\mathrm{d}\tau, b(\Pi_1\nabla v_h)\cdot\boldsymbol{n}\right\rangle_{\partial\Omega_h} \\
\lesssim & \frac{1}{\Delta t}\sum_{e\in\mathcal{E}_h^b} \big\|\Pi_1^{E_e}\nabla v_h\big\|_e \int_{t^{n-1}}^{t^n} \big\|\hat g_t(\tau) - \big(u_t(\tau) + \rho\nabla u_t(\tau)\cdot\boldsymbol{n}\big)\big\|_e\mathrm{d}\tau \\
\lesssim & \frac{1}{\Delta t}\sum_{e\in\mathcal{E}_h^b} h^4 h_e^{\frac12}\big\|\Pi_1^{E_e}\nabla v_h\big\|_e \int_{t^{n-1}}^{t^n} |u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau \\
\lesssim & h^3\|v_h\|_{\Omega_h}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} |u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau.
\end{aligned}
\tag{4.22}
$$

Similarly, employing (4.2) further, it follows that

$$
\begin{aligned}
&\left\langle \frac{1}{\Delta t}\Big(\hat g(t^n) - \hat g(t^{n-1}) - \big(u(t^n) - u(t^{n-1}) + \rho\nabla\big(u(t^n) - u(t^{n-1})\big)\cdot \boldsymbol{n}\big)\Big), \gamma\bar h^{-1}v_h\right\rangle_{\partial\Omega_h} \\
= & \frac{1}{\Delta t}\left\langle \int_{t^{n-1}}^{t^n} \hat g_t(\tau) - \big(u_t(\tau) + \rho\nabla u_t(\tau)\cdot\boldsymbol{n}\big)\mathrm{d}\tau, \gamma\bar h^{-1}v_h\right\rangle_{\partial\Omega_h}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \frac{1}{\Delta t}\sum_{e\in\mathcal{E}_h^b} h_e^{-1}\|v_h\|_e \int_{t^{n-1}}^{t^n} \big\|\hat{g}_t(\tau)-\big(u_t(\tau)+\rho\nabla u_t(\tau)\cdot\boldsymbol{n}\big)\big\|_e \mathrm{d}\tau \\
&\lesssim \frac{1}{\Delta t}\sum_{e\in\mathcal{E}_h^b} h^4\big(h_{E_e}^{-1}\|v_h\|_{E_e}+|v_h|_{1,E_e}\big)\int_{t^{n-1}}^{t^n} |u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau \\
&\lesssim h^3\|v_h\|_{\Omega_h}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}|u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau. \qquad (4.23)
\end{aligned}
$$

Eqs. (4.22) and (4.23) give the desired result (4.20). The proof for (4.21) can be obtained by a similar analysis to (4.20). □

Theorem 4.2. *Let u be the solution of (1.1) and $\{u_h^n\}_{n=1}^{N_T}$ be the solution of (3.32). Suppose that $u_t \in L^2[0,T;H^3(\Omega)\cap W^{2,\infty}(\Omega)]$, $u_{ttt}\in L^2[0,T;H^2(\Omega)]$ and $u_0\in H^3(\Omega)$. Assume further that for $n=1,2,\ldots,N_T$, there is a positive constant $C_{\max}$ such that*

$$
\max_{0\le i\le n}\Big\{\|f(t^i)\|_{3,\Omega},\big\|c\big(u(t^i)\big)\big\|_{3,\Omega},\|u(t^i)\|_{3,\Omega},|u(t^i)|_{2,\infty,\Omega}\Big\}\le C_{\max},
$$

then for $\Delta t\lesssim 1$, the following error estimates hold:

$$
\begin{aligned}
\|u(t^n)-u_h^n\|_{\Omega_h} \lesssim h^3\Big(&\|u_t\|_{L^2[0,t^n;H^3(\Omega)]}+\|u_t\|_{L^2[0,t^n;W^{2,\infty}(\Omega)]}+\max_{0\le i\le n}\|f(t^i)\|_{3,\Omega} \\
&+\max_{0\le i\le n}\big\|c\big(u(t^i)\big)\big\|_{3,\Omega}+\max_{0\le i\le n}\|u(t^i)\|_{3,\Omega}+\max_{0\le i\le n}|u(t^i)|_{2,\infty,\Omega}\Big) \\
&+(\Delta t)^2\|u_{ttt}\|_{L^2[0,t^n;H^2(\Omega)]}, \qquad (4.24)\\
\|u(t^n)-u_h^n\|_{\mathcal{N}} \lesssim h^3\Big(&\|u_t\|_{L^2[0,t^n;H^3(\Omega)]}+\|u_t\|_{L^2[0,t^n;W^{2,\infty}(\Omega)]}+\max_{0\le i\le n}\|f(t^i)\|_{3,\Omega} \\
&+\max_{0\le i\le n}\big\|c\big(u(t^i)\big)\big\|_{3,\Omega}+\max_{0\le i\le n}\|u(t^i)\|_{3,\Omega}+\max_{0\le i\le n}|u(t^i)|_{2,\infty,\Omega}\Big) \\
&+(\Delta t)^2\|u_{ttt}\|_{L^2[0,t^n;H^2(\Omega)]}+h^2\big(\|u_0\|_{3,\Omega}+\|u(t^n)\|_{3,\Omega}\big). \qquad (4.25)
\end{aligned}
$$

Proof. Using the energy projection $\mathtt{P}$, the error $u(t^n)-u_h^n$ can be decomposed as

$$
u(t^n)-u_h^n = u(t^n)-\mathtt{P}u(t^n)+\mathtt{P}u(t^n)-u_h^n.
$$

Due to Lemmas 4.2 and 4.3, we obtain

$$
\|u(t^n)-\mathtt{P}u(t^n)\|_{\mathcal{N}}\lesssim h^2\|u(t^n)\|_{3,\Omega},\quad n=1,2,\ldots,N_T, \qquad (4.26)
$$

$$
\|u(t^n)-\mathtt{P}u(t^n)\|_{\Omega_h}\lesssim h^3\|u(t^n)\|_{3,\Omega},\quad n=1,2,\ldots,N_T. \qquad (4.27)
$$

Set $\theta^n=\mathtt{P}u(t^n)-u_h^n$. For the convenience of subsequent analysis, we introduce the following notations:

$$
\mathring{\eta}^n := \frac{\eta^n-\eta^{n-1}}{\Delta t},\quad \bar{\eta}^n := \frac{\eta^n+\eta^{n-1}}{2},
$$

where $\eta\in\{u,u_t,\theta,f,\hat{g}\}$ and $\eta^n=\eta(t^n)$ for $\eta\in\{u,u_t,f,\hat{g}\}$.

On the basis of the definition (4.1) of $\mathtt{P}$ and the fully discrete scheme (3.32), a straightforward calculation yields

$$
\mathcal{A}_h\big(\mathring{\theta}^n,v_h\big)+\mathcal{N}_h\big(\mathring{\theta}^n,v_h\big)+\mathcal{N}_h\big(\bar{\theta}^n,v_h\big)
$$

$$
\begin{aligned}
&= \mathcal{A}_h\big(\mathtt{P}\mathring{u}^n, v_h\big) + \mathcal{N}_h\big(\mathtt{P}\mathring{u}^n, v_h\big) + \mathcal{N}_h\big(\mathtt{P}\bar{u}^n, v_h\big) \\
&\quad + \frac{1}{2}\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h^n) + \mathcal{I}_{\mathcal{S}}c(u_h^{n-1}), v_h\big) - \mathcal{L}_{\mathcal{N}}^n(v_h) \\
&= \mathcal{A}_h(\mathtt{P}\mathring{u}^n, v_h) - \big(\nabla\cdot(b\nabla\mathring{u}^n), v_h\big)_{\Omega_h} - \big(\nabla\cdot(b\nabla\bar{u}^n), v_h\big)_{\Omega_h} \\
&\quad + \big\langle \mathring{g}^n - (\mathring{u}^n + \rho\nabla\mathring{u}^n\cdot\boldsymbol{n})\,, b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h} \\
&\quad + \big\langle \bar{g}^n - (\bar{u}^n + \rho\nabla\bar{u}^n\cdot\boldsymbol{n}), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h} \\
&\quad - \frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E \big(\Pi_0^E f(t^n) + \Pi_0^E f(t^{n-1})\big)v_h\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{1}{2}\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c(u_h^n) + \mathcal{I}_{\mathcal{S}}c(u_h^{n-1}), v_h\big).
\end{aligned}
$$

For the solution u of the problem (1.1), it can easily be verified that

$$
\begin{aligned}
&(a\bar{u}_t^n, v_h)_{\Omega_h} - \big(\nabla\cdot(b\nabla\bar{u}_t^n), v_h\big)_{\Omega_h} - \big(\nabla\cdot(b\nabla\bar{u}^n), v_h\big)_{\Omega_h} \\
&\quad + \frac{1}{2}\big(c(u^n) + c(u^{n-1}), v_h\big)_{\Omega_h} = (\bar{f}^n, v_h)_{\Omega_h},
\end{aligned}
$$

then we obtain

$$
\mathcal{A}_h\big(\mathring{\theta}^n, v_h\big) + \mathcal{N}_h\big(\mathring{\theta}^n, v_h\big) + \mathcal{N}_h\big(\bar{\theta}^n, v_h\big) = \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \mathcal{Y}_4 + \mathcal{Y}_5 + \mathcal{Y}_6,
$$

where

$$
\begin{aligned}
\mathcal{Y}_1 &= \mathcal{A}_h\big(\mathtt{P}\mathring{u}^n, v_h\big) - \big(a\bar{u}_t^n, v_h\big)_{\Omega_h}, \\
\mathcal{Y}_2 &= \big(\nabla\cdot\big(b\nabla(\bar{u}_t^n - \mathring{u}^n)\big), v_h\big)_{\Omega_h}, \\
\mathcal{Y}_3 &= \big(\bar{f}^n, v_h\big)_{\Omega_h} - \frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E \big(\Pi_0^E f^n + \Pi_0^E f^{n-1}\big)v_h\mathrm{d}\boldsymbol{x}, \\
\mathcal{Y}_4 &= \frac{1}{2}\mathcal{C}_h\big(\mathcal{I}_{\mathcal{S}}c\big(u_h^n\big) + \mathcal{I}_{\mathcal{S}}c\big(u_h^{n-1}\big), v_h\big) - \frac{1}{2}\big(c(u^n) + c(u^{n-1}), v_h\big)_{\Omega_h}, \\
\mathcal{Y}_5 &= \big\langle \mathring{g}^n - (\mathring{u}^n + \rho\nabla\mathring{u}^n\cdot\boldsymbol{n}), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}, \\
\mathcal{Y}_6 &= \big\langle \bar{g}^n - (\bar{u}^n + \rho\nabla\bar{u}^n\cdot\boldsymbol{n}), b(\Pi_1\nabla v_h)\cdot\boldsymbol{n} - \gamma\bar{h}^{-1}v_h\big\rangle_{\partial\Omega_h}.
\end{aligned}
$$

From (3.5), (3.8) and (4.5), we have

$$
\begin{aligned}
\mathcal{Y}_1 &= \sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E\big(\mathtt{P}\mathring{u}^n, v_h\big) - \big(a\bar{u}_t^n, v_h\big)_E \\
&= \sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E\big(\mathtt{P}\mathring{u}^n - \Pi_0^E\mathring{u}^n, v_h\big) + \big(a\Pi_0^E\mathring{u}^n, \Pi_0^E v_h\big)_E - \big(a\bar{u}_t^n, v_h\big)_E \\
&= \sum_{E\in\mathcal{T}_h}\mathcal{A}_h^E\big(\mathtt{P}\mathring{u}^n - \Pi_0^E\mathring{u}^n, v_h\big) + \big(a\Pi_0^E\mathring{u}^n - a\mathring{u}^n, \Pi_0^E v_h\big)_E - \big(a\bar{u}_t^n - \Pi_0^E(a\mathring{u}^n), v_h\big)_E \\
&\lesssim \big(h^3\|\mathring{u}^n\|_{3,\Omega} + \|\mathring{u}^n - \bar{u}_t^n\|_{\Omega_h}\big)\|v_h\|_{\Omega_h}.
\end{aligned}
$$

It is easy to see that the term $\|\mathring{u}^n\|_{3,\Omega}$ can be bounded as

$$
\|\mathring{u}^n\|_{3,\Omega} = \frac{1}{\Delta t}\left\|\int_{t^{n-1}}^{t^n} u_t(\tau)\mathrm{d}\tau\right\|_{3,\Omega} \lesssim \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t(\tau)\|_{3,\Omega}\mathrm{d}\tau,
$$

and we can estimate the term $\|\mathring{u}^n - \bar{u}_t^n\|_{\Omega_h}$ as

$$\begin{aligned}\left\|\mathring{u}^n - \bar{u}_t^n\right\|_{\Omega_h} &= \frac{1}{2\Delta t}\left\|2u^n - 2u^{n-1} - \Delta t u_t^n - \Delta t u_t^{n-1}\right\|_{\Omega_h} \\ &\le \frac{1}{\Delta t}\left\|u^n - u^{n-1} - \Delta t u_t\left(\frac{1}{2}(t^n + t^{n-1})\right)\right\|_{\Omega_h} \\ &\quad + \left\|u_t\left(\frac{1}{2}(t^n + t^{n-1})\right) - \frac{1}{2}u_t^n - \frac{1}{2}u_t^{n-1}\right\|_{\Omega_h} \\ &= \frac{1}{2\Delta t}\left\|\int_{t^{n-1}}^{\frac{t^n+t^{n-1}}{2}} (\tau - t^{n-1})^2 u_{ttt}(\tau)\mathrm{d}\tau + \int_{\frac{t^n+t^{n-1}}{2}}^{t^n} (\tau - t^n)^2 u_{ttt}(\tau)\mathrm{d}\tau\right\|_{\Omega_h} \\ &\quad + \frac{1}{2}\left\|\int_{\frac{t^n+t^{n-1}}{2}}^{t^n} (\tau - t^n) u_{ttt}(\tau)\mathrm{d}\tau - \int_{t^{n-1}}^{\frac{t^n+t^{n-1}}{2}} (\tau - t^{n-1}) u_{ttt}(\tau)\mathrm{d}\tau\right\|_{\Omega_h} \\ &\lesssim \Delta t \int_{t^{n-1}}^{t^n} \|u_{ttt}(\tau)\|_{\Omega}\mathrm{d}\tau.\end{aligned}$$

Therefore,

$$\mathcal{Y}_1 \lesssim \frac{1}{\Delta t}\left(h^3 \int_{t^{n-1}}^{t^n} \|u_t(\tau)\|_{3,\Omega}\mathrm{d}\tau + (\Delta t)^2 \int_{t^{n-1}}^{t^n} \|u_{ttt}(\tau)\|_{\Omega}\mathrm{d}\tau\right)\|v_h\|_{\Omega_h}. \tag{4.28}$$

Similar to the estimate for $\|\mathring{u}^n - \bar{u}_t^n\|_{\Omega_h}$, the following estimation for $\mathcal{Y}_2$ can be obtained:

$$\mathcal{Y}_2 \lesssim \Delta t \|v_h\|_{\Omega_h} \int_{t^{n-1}}^{t^n} \|u_{ttt}(\tau)\|_{2,\Omega}\mathrm{d}\tau. \tag{4.29}$$

By (3.8), $\mathcal{Y}_3$ can be simply bounded as

$$\mathcal{Y}_3 \lesssim h^3 \|v_h\|_{\Omega_h} \max_{n-1\le i\le n} \|f^i\|_{3,\Omega}. \tag{4.30}$$

Employing [18, Theorem 1] again, we have the following estimate for $\mathcal{Y}_4$:

$$\mathcal{Y}_4 \lesssim \Big(h^3 \max_{n-1\le i\le n} \|c(u^i)\|_{3,\Omega} + h^3 \max_{n-1\le i\le n} \|u^i\|_{3,\Omega} + \|\theta^n\|_{\Omega_h} + \|\theta^{n-1}\|_{\Omega_h}\Big)\|v_h\|_{\Omega_h}. \tag{4.31}$$

The error terms $\mathcal{Y}_5$ and $\mathcal{Y}_6$ result from the deformation of $\partial\Omega$ to $\partial\Omega_h$ and their estimates have been established in Lemma 4.5. Combining (4.28)-(4.31), we have

$$\begin{aligned}&\mathcal{A}_h\big(\mathring{\theta}^n, v_h\big) + \mathcal{N}_h\big(\mathring{\theta}^n, v_h\big) + \mathcal{N}_h\big(\bar{\theta}^n, v_h\big) \\ &\lesssim \frac{1}{\Delta t}\left(h^3\big(\mathcal{Z}_1^n + \mathcal{Z}_2^n\big) + (\Delta t)^2\big(\mathcal{Z}_3^n + \mathcal{Z}_4^n\big) + h^3\Delta t\big(\mathcal{Z}_5^n + \mathcal{Z}_6^n + \mathcal{Z}_7^n + \mathcal{Z}_8^n\big)\right)\|v_h\|_{\Omega_h} \\ &\quad + \big(\|\theta^n\|_{\Omega_h} + \|\theta^{n-1}\|_{\Omega_h}\big)\|v_h\|_{\Omega_h},\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}\mathcal{Z}_1^n &= \int_{t^{n-1}}^{t^n} \|u_t(\tau)\|_{3,\Omega}\mathrm{d}\tau, & \mathcal{Z}_2^n &= \int_{t^{n-1}}^{t^n} |u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau, \\ \mathcal{Z}_3^n &= \int_{t^{n-1}}^{t^n} \|u_{ttt}(\tau)\|_{\Omega}\mathrm{d}\tau, & \mathcal{Z}_4^n &= \int_{t^{n-1}}^{t^n} \|u_{ttt}(\tau)\|_{2,\Omega}\mathrm{d}\tau, \\ \mathcal{Z}_5^n &= \max_{n-1\le i\le n} \|f^i\|_{3,\Omega}, & \mathcal{Z}_6^n &= \max_{n-1\le i\le n} \|c(u^i)\|_{3,\Omega}, \\ \mathcal{Z}_7^n &= \max_{n-1\le i\le n} \|u^i\|_{3,\Omega}, & \mathcal{Z}_8^n &= \max_{n-1\le i\le n} |u^i|_{2,\infty,\Omega}.\end{aligned}$$

Taking v_h in (4.32) as $\mathring{\theta}^n + \bar{\theta}^n$ and utilizing the coercivity of $\mathcal{A}_h(\cdot\,,\cdot)$ and $\mathcal{N}_h(\cdot\,,\cdot)$, we arrive at

$$\begin{aligned}
&\mathcal{A}_h(\theta^n,\theta^n) - \mathcal{A}_h(\theta^{n-1},\theta^{n-1})\\
&\lesssim \frac{1}{\Delta t}\Big(h^6\Big(\big(\mathcal{Z}_1^n\big)^2 + \big(\mathcal{Z}_2^n\big)^2\Big) + (\Delta t)^4\Big(\big(\mathcal{Z}_3^n\big)^2 + \big(\mathcal{Z}_4^n\big)^2\Big)\\
&\qquad + h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^n\big)^2 + \big(\mathcal{Z}_6^n\big)^2 + \big(\mathcal{Z}_7^n\big)^2 + \big(\mathcal{Z}_8^n\big)^2\Big)\Big)\\
&\quad + \Delta t\big(\|\theta^n\|^2_{\Omega_h} + \|\theta^{n-1}\|^2_{\Omega_h}\big). \qquad (4.33)
\end{aligned}$$

A repeated application of (4.33) gives

$$\begin{aligned}
\|\theta^n\|^2_{\Omega_h} \lesssim\ & \frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6\Big(\big(\mathcal{Z}_1^i\big)^2 + \big(\mathcal{Z}_2^i\big)^2\Big) + (\Delta t)^4\Big(\big(\mathcal{Z}_3^i\big)^2 + \big(\mathcal{Z}_4^i\big)^2\Big)\Big)\\
&+ \frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^i\big)^2 + \big(\mathcal{Z}_6^i\big)^2 + \big(\mathcal{Z}_7^i\big)^2 + \big(\mathcal{Z}_8^i\big)^2\Big)\Big)\\
&+ \Delta t\sum_{i=0}^{n}\|\theta^i\|^2_{\Omega_h} + \|\theta^0\|^2_{\Omega_h},
\end{aligned}$$

then for $\Delta t \lesssim 1$ (which is also required in [20, Theorem 2.1]), applying the discrete Grönwall's lemma, (3.11) and (4.5) yields

$$\begin{aligned}
\|\theta^n\|^2_{\Omega_h} \lesssim\ & \frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6\Big(\big(\mathcal{Z}_1^i\big)^2 + \big(\mathcal{Z}_2^i\big)^2\Big) + (\Delta t)^4\Big(\big(\mathcal{Z}_3^i\big)^2 + \big(\mathcal{Z}_4^i\big)^2\Big)\Big)\\
&+ \frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^i\big)^2 + \big(\mathcal{Z}_6^i\big)^2 + \big(\mathcal{Z}_7^i\big)^2 + \big(\mathcal{Z}_8^i\big)^2\Big)\Big) + h^6\|u_0\|^2_{3,\Omega}. \qquad (4.34)
\end{aligned}$$

According to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{i=1}^{n} h^6\Big(\big(\mathcal{Z}_1^i\big)^2 + \big(\mathcal{Z}_2^i\big)^2\Big)\\
&\lesssim \frac{1}{\Delta t}\sum_{i=1}^{n} h^6\bigg(\Big(\int_{t^{i-1}}^{t^i}\|u_t(\tau)\|_{3,\Omega}\mathrm{d}\tau\Big)^2 + \Big(\int_{t^{i-1}}^{t^i}|u_t(\tau)|_{2,\infty,\Omega}\mathrm{d}\tau\Big)^2\bigg)\\
&\lesssim h^6\left(\|u_t\|^2_{L^2[0,t^n;H^3(\Omega)]} + \|u_t\|^2_{L^2[0,t^n;W^{2,\infty}(\Omega)]}\right), \qquad (4.35)\\
&\frac{1}{\Delta t}\sum_{i=1}^{n}(\Delta t)^4\Big(\big(\mathcal{Z}_3^i\big)^2 + \big(\mathcal{Z}_4^i\big)^2\Big)\\
&= \frac{1}{\Delta t}\sum_{i=1}^{n}(\Delta t)^4\bigg(\Big(\int_{t^{i-1}}^{t^i}\|u_{ttt}(\tau)\|_{\Omega}\mathrm{d}\tau\Big)^2 + \Big(\int_{t^{i-1}}^{t^i}\|u_{ttt}(\tau)\|_{2,\Omega}\mathrm{d}\tau\Big)^2\bigg)\\
&\lesssim (\Delta t)^4\|u_{ttt}\|^2_{L^2[0,t^n;H^2(\Omega)]}, \qquad (4.36)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{i=1}^{n} h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^i\big)^2 + \big(\mathcal{Z}_6^i\big)^2 + \big(\mathcal{Z}_7^i\big)^2 + \big(\mathcal{Z}_8^i\big)^2\Big)\\
&\lesssim t^n h^6\Big(\max_{0\le i\le n}\|f^i\|^2_{3,\Omega} + \max_{0\le i\le n}\|c(u^i)\|^2_{3,\Omega} + \max_{0\le i\le n}\|u^i\|^2_{3,\Omega} + \max_{0\le i\le n}|u^i|^2_{2,\infty,\Omega}\Big). \qquad (4.37)
\end{aligned}$$

Substituting (4.35)-(4.37) into (4.34) and noticing (4.27), we can obtain (4.24).

Replacing v_h in (4.32) with $\Delta t\mathring{\theta}^n$ and using the coercivity of $\mathcal{A}_h(\cdot,\cdot)$ and $\mathcal{N}_h(\cdot,\cdot)$, we have

$$\begin{aligned}
&\mathcal{N}_h(\theta^n,\theta^n)-\mathcal{N}_h(\theta^{n-1},\theta^{n-1})\\
&\lesssim \frac{1}{\Delta t}\Big(h^6\Big(\big(\mathcal{Z}_1^n\big)^2+\big(\mathcal{Z}_2^n\big)^2\Big)+(\Delta t)^4\Big(\big(\mathcal{Z}_3^n\big)^2+\big(\mathcal{Z}_4^n\big)^2\Big)\\
&\qquad +h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^n\big)^2+\big(\mathcal{Z}_6^n\big)^2+\big(\mathcal{Z}_7^n\big)^2+\big(\mathcal{Z}_8^n\big)^2\Big)\Big)\\
&\quad +\|\theta^n\|_{\Omega_h}^2+\|\theta^{n-1}\|_{\Omega_h}^2.
\end{aligned}\tag{4.38}$$

By repeated application of (4.38), it follows that

$$\begin{aligned}
\|\theta^n\|_{\mathcal{N}}^2 \lesssim\; & \frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6\Big(\big(\mathcal{Z}_1^i\big)^2+\big(\mathcal{Z}_2^i\big)^2\Big)+(\Delta t)^4\Big(\big(\mathcal{Z}_3^i\big)^2+\big(\mathcal{Z}_4^i\big)^2\Big)\Big)\\
&+\frac{1}{\Delta t}\sum_{i=1}^{n}\Big(h^6(\Delta t)^2\Big(\big(\mathcal{Z}_5^i\big)^2+\big(\mathcal{Z}_6^i\big)^2+\big(\mathcal{Z}_7^i\big)^2+\big(\mathcal{Z}_8^i\big)^2\Big)\Big)\\
&+\sum_{i=0}^{n}\|\theta^i\|_{\Omega_h}^2+\|\theta^0\|_{\mathcal{N}}^2,
\end{aligned}$$

then, from (4.34)-(4.37), we get

$$\begin{aligned}
\|\theta^n\|_{\mathcal{N}} \lesssim\; & h^3\Big(\|u_t\|_{L^2[0,t^n;H^3(\Omega)]}+\|u_t\|_{L^2[0,t^n;W^{2,\infty}(\Omega)]}+\max_{0\le i\le n}\|f(t^i)\|_{3,\Omega}\\
&+\max_{0\le i\le n}\big\|c\big(u(t^i)\big)\big\|_{3,\Omega}+\max_{0\le i\le n}\|u(t^i)\|_{3,\Omega}+\max_{0\le i\le n}|u(t^i)|_{2,\infty,\Omega}\Big)\\
&+\|\mathtt{P}u_0-u_0\|_{\mathcal{N}}+\|u_0-\mathcal{I}_{\mathcal{S}}u_0\|_{\mathcal{N}}+(\Delta t)^2\|u_{ttt}\|_{L^2[0,t^n;H^2(\Omega)]}.
\end{aligned}$$

Now, (4.25) can be derived according to Lemma 4.2 and (4.26). □

5. Numerical Experiments

In this section, four numerical examples are provided to verify the theoretical results in the case of different nonlinear terms and curved domains. The first one is used to show the convergence rate of Crank-Nicolson scheme. The second one is used to confirm the convergence rate of the spatial approximation based on Nitsche-based projection SVEM. Both convergence rates are verified in the third example. The last one is for the case of using meshes with hanging nodes. In all experiments, we select the tolerance for the stopping criterion of Newton's iteration as 10^{-6} and we set the penalty parameter γ as 500. The final time T is all set to be 1. We employ the following approximation errors:

$$\mathtt{E}_1 := \left(\sum_{E\in\mathcal{T}_h}\big|u(T)-\Pi_0^E u_h^{N_T}\big|_{1,E}^2+\sum_{e\in\mathcal{E}_h^b}h_e^{-1}\big\|u(T)-u_h^{N_T}\big\|_e^2\right)^{\frac{1}{2}},$$

$$\mathtt{E}_0 := \left(\sum_{E\in\mathcal{T}_h}\big\|u(T)-\Pi_0^E u_h^{N_T}\big\|_E^2\right)^{\frac{1}{2}}.$$

According to Lemma 3.1, [10, Lemma 2.17] and Theorem 4.2, it is easy to check that

$$\mathtt{E}_1 \lesssim (\Delta t)^2+h^2, \tag{5.1}$$

$$\mathtt{E}_0 \lesssim (\Delta t)^2+h^3. \tag{5.2}$$

So, $\mathtt{E}_1$ and $\mathtt{E}_0$ converge with the same orders as $\|u(t^n)-u_h^n\|_{\mathcal{N}}$ and $\|u(t^n)-u_h^n\|_{\Omega_h}$ respectively.

Example 5.1. Let $a(\boldsymbol{x}) = x + y + 3$ and $b(\boldsymbol{x}) = x + y + 2$. Consider the following initial-boundary value problem of semilinear pseudo-parabolic equation:

$$\begin{aligned} &a(\boldsymbol{x})u_t - \nabla \cdot \big(b(\boldsymbol{x})\nabla u_t + b(\boldsymbol{x})\nabla u\big) + u^2 = f(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ &u = g(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega, \end{aligned}$$

where

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, \ -\sin\big(x(1-x)\big) \le y \le 1 + \sin\big(x(1-x)\big)\right\},$$

and the data f, g and u_0 are chosen in such a way that the exact solution is $u = \exp(-t)(x^3+y^3)$.

Two types of meshes (denoted as $\texttt{MESH}_1$ and $\texttt{MESH}_2$) are employed in this example and the schematic diagrams are provided in Fig. 5.1. In Table 5.1, for different Δt, we report the approximation errors $\texttt{E}_1$ and $\texttt{E}_0$ and calculate the orders of convergence. Due to the relatively small mesh size, we predict the temporal error dominates and the results listed in Table 5.1 confirm our prediction, i.e. the second-order convergence of the Crank-Nicolson scheme is observed. Thus, the convergence order for the temporal discretization derived in Theorem 4.2 is illustrated.

Table 5.1: Errors $\texttt{E}_1$ and $\texttt{E}_0$ and the temporal convergence orders for Example 5.1 with fixed h.

$\texttt{MESH}_1$ (with 2500 elements)					$\texttt{MESH}_2$ (with 2500 elements)				
Δt	$\texttt{E}_1$	Order	$\texttt{E}_0$	Order	Δt	$\texttt{E}_1$	Order	$\texttt{E}_0$	Order
1/4	7.5328e-02		1.7830e-03		1/4	6.2723e-02		1.7694e-03	
1/6	3.3353e-02	2.0093	7.8947e-04	2.0093	1/6	2.7772e-02	2.0093	7.8344e-04	2.0093
1/8	1.8736e-02	2.0046	4.4349e-04	2.0046	1/8	1.5601e-02	2.0045	4.4011e-04	2.0046
1/10	1.1984e-02	2.0027	2.8366e-04	2.0027	1/10	9.9791e-03	2.0026	2.8150e-04	2.0027

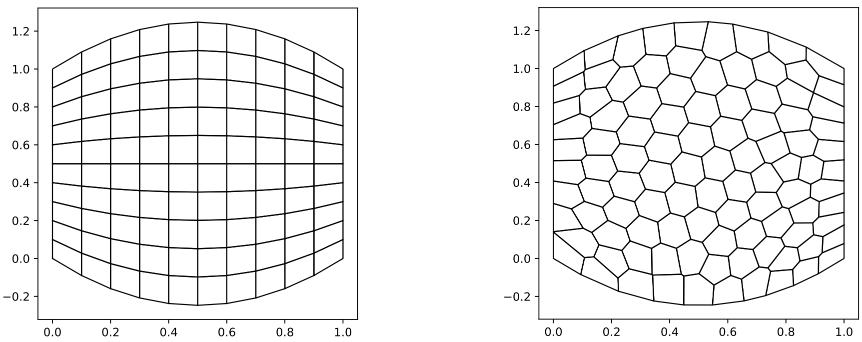

Fig. 5.1. $\texttt{MESH}_1$ (left) and $\texttt{MESH}_2$ (right) used in Example 5.1.

Example 5.2. Let $a(\boldsymbol{x}) = x^2 + y^2 + 1$ and $b(\boldsymbol{x}) = x^2 + y^2 + 5$. Take into account the following initial-boundary value problem of semilinear pseudo-parabolic equation:

$$\begin{aligned} &a(\boldsymbol{x})u_t - \nabla \cdot \big(b(\boldsymbol{x})\nabla u_t + b(\boldsymbol{x})\nabla u\big) + u(1-u) = f(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ &u = g(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega, \end{aligned}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. The data f, g and u_0 are obtained by exact solution

$$u = \exp(-t)\big(x + y + \sin(2\pi x)\sin(2\pi y)\big).$$

Two kinds of meshes (denoted as $\texttt{MESH}_3$ and $\texttt{MESH}_4$) are employed in this test and we show their schematic diagrams in Fig. 5.2. The time step size Δt is set to be 1/1000 such that temporal error is small and the spatial error dominates. In Table 5.2, we list the orders of convergence for $\texttt{E}_1$ and $\texttt{E}_0$. Therefore, the orders of convergence in the spatial direction derived in Theorem 4.2 are illustrated.

Table 5.2: Errors $\texttt{E}_1$ and $\texttt{E}_0$ and the spatial convergence orders for Example 5.2 with $\Delta t = 1/1000$.

$\texttt{MESH}_3$					$\texttt{MESH}_4$				
h	$\texttt{E}_1$	Order	$\texttt{E}_0$	Order	h	$\texttt{E}_1$	Order	$\texttt{E}_0$	Order
1.9858e-01	2.0970e-01		5.0590e-03		2.8707e-01	4.1617e-01		1.5184e-02	
1.4950e-01	1.1802e-01	2.0249	2.1133e-03	3.0749	2.3845e-01	2.8809e-01	1.9823	8.5620e-03	3.0873
1.0144e-01	5.4187e-02	2.0071	6.4253e-04	3.0700	1.2924e-01	8.5982e-02	1.9740	1.3604e-03	3.0033
6.8146e-02	2.4095e-02	2.0372	1.8756e-04	3.0951	7.7051e-02	2.8637e-02	2.1258	2.5934e-04	3.2047

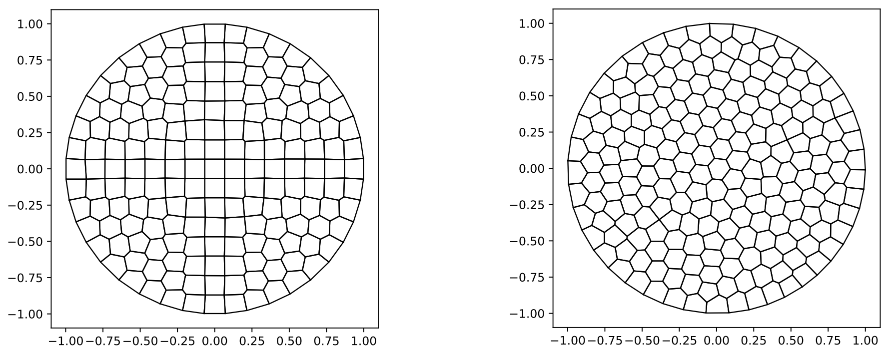

Fig. 5.2. $\texttt{MESH}_3$ (left) and $\texttt{MESH}_4$ (right) used in Example 5.2.

Example 5.3. Consider the following semilinear pseudo-parabolic equation:

$$\begin{aligned} & u_t - \Delta(u_t + u) + u^3 = f(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ & u = g(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \\ & u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega, \end{aligned}$$

where

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\frac{1}{3}x(1 - x^2) \leq y \leq 1 + \frac{1}{3}x(1 - x^2) \right\},$$

and the data f, g and u_0 are gotten by exact solution

$$u = \exp(-t)(x^5 + y^5 + 4x^3 y + 2xy + 5).$$

The meshes (denoted as $\texttt{MESH}_5$ and $\texttt{MESH}_6$) employed are illustrated in Fig. 5.3. Similar to Examples 5.1 and 5.2, to verify the convergence in temporal (spatial) direction, mesh size

h (time step size Δt) is chosen to be small. In Table 5.3, it is easy to see that the second-order convergence of the Crank-Nicolson scheme is observed. In Table 5.4, the second-order convergence for $\mathtt{E}_1$ and third-order convergence for $\mathtt{E}_0$ are also shown. Hence, the predictions (5.1) and (5.2) are confirmed and the theoretical conclusions in Theorem 4.2 are verified.

Table 5.3: Errors $\mathtt{E}_1$ and $\mathtt{E}_0$ and the temporal convergence orders for Example 5.3 with fixed h.

$\mathtt{MESH}_5$ (with 5000 elements)					$\mathtt{MESH}_6$ (with 2304 elements)				
Δt	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order	Δt	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order
1/4	3.6826e-01		1.3706e-02		1/4	3.3518e-01		1.3681e-02	
1/8	9.1594e-02	2.0074	3.4089e-03	2.0074	1/8	8.3371e-02	2.0073	3.4029e-03	2.0074
1/12	4.0672e-02	2.0022	1.5136e-03	2.0023	1/12	3.7025e-02	2.0019	1.5110e-03	2.0022
1/16	2.2872e-02	2.0008	8.5115e-04	2.0011	1/16	2.0827e-02	2.0000	8.4972e-04	2.0009

Table 5.4: Errors $\mathtt{E}_1$ and $\mathtt{E}_0$ and the spatial convergence orders for Example 5.3 with $\Delta t = 1/1000$.

$\mathtt{MESH}_5$					$\mathtt{MESH}_6$				
h	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order	h	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order
1.8614e-01	9.1154e-03		1.0694e-04		2.3857e-01	2.2539e-02		5.1535e-04	
1.2587e-01	4.0697e-03	2.0610	3.1698e-05	3.1078	1.2350e-01	4.6503e-03	2.3970	5.3287e-05	3.4462
9.5083e-02	2.2929e-03	2.0454	1.3481e-05	3.0481	8.3264e-02	1.9664e-03	2.1834	1.4873e-05	3.2373
7.6397e-02	1.4685e-03	2.0364	6.8830e-06	3.0722	6.2814e-02	1.0811e-03	2.1224	6.0664e-06	3.1819

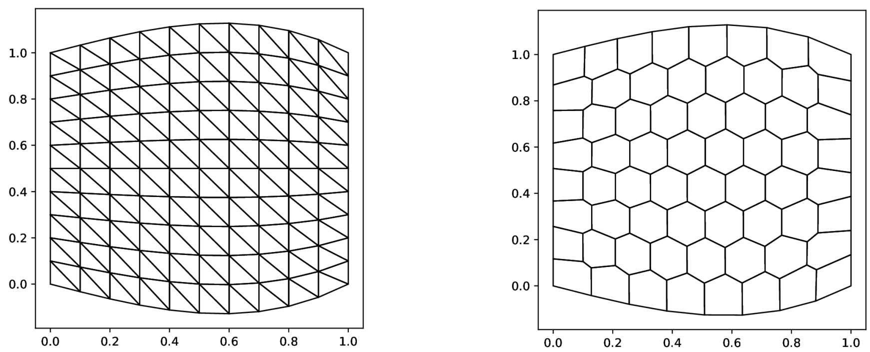

Fig. 5.3. $\mathtt{MESH}_5$ (left) and $\mathtt{MESH}_6$ (right) used in Example 5.3.

Finally, since VEM can deal with meshes with hanging nodes straightforwardly [24], we give an example to show its effectiveness.

Example 5.4. Let $a(\boldsymbol{x}), b(\boldsymbol{x}), \Omega, c(u)$ and the exact solution u be the same as Example 5.1.

Two types of meshes with hanging nodes (denoted as $\mathtt{MESH}_7$ and $\mathtt{MESH}_8$) are employed in this example and their schematic diagrams are plotted in Fig. 5.4. The numerical results are listed in Tables 5.5 and 5.6. We observe that $\mathtt{E}_1 = O((\Delta t)^2 + h^2)$ and $\mathtt{E}_0 = O((\Delta t)^2 + h^3)$, thus the theoretical results established in Theorem 4.2 are confirmed. This reveals that VEM has advantages over finite element methods in handling meshes with hanging nodes.

Table 5.5: Errors $\mathtt{E}_1$ and $\mathtt{E}_0$ and the temporal convergence orders for Example 5.4 with fixed h.

$\mathtt{MESH}_7$ (with 1740 elements)					$\mathtt{MESH}_8$ (with 2500 elements)				
Δt	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order	Δt	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order
1/4	5.4876e-02		1.7751e-03		1/4	5.8071e-02		1.7623e-03	
1/6	2.4296e-02	2.0094	7.8595e-04	2.0093	1/6	2.5712e-02	2.0093	7.8030e-04	2.0093
1/8	1.3648e-02	2.0046	4.4152e-04	2.0045	1/8	1.4444e-02	2.0045	4.3835e-04	2.0046
1/10	8.7302e-03	2.0024	2.8241e-04	2.0027	1/10	9.2393e-03	2.0025	2.8037e-04	2.0027

Table 5.6: Errors $\mathtt{E}_1$ and $\mathtt{E}_0$ and the spatial convergence orders for Example 5.4 with $\Delta t = 1/1000$.

$\mathtt{MESH}_7$					$\mathtt{MESH}_8$				
h	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order	h	$\mathtt{E}_1$	Order	$\mathtt{E}_0$	Order
1.8161e-01	2.1438e-03		4.1281e-05		2.3020e-01	2.5748e-03		5.4863e-05	
1.1814e-01	9.1139e-04	1.9891	1.1451e-05	2.9820	1.1989e-01	6.8566e-04	2.0283	7.6643e-06	3.0172
8.7552e-02	5.0221e-04	1.9891	4.6874e-06	2.9813	9.8083e-02	4.5591e-04	2.0325	4.2204e-06	2.9718
6.9560e-02	3.1762e-04	1.9917	2.3586e-06	2.9858	8.3437e-02	3.3253e-04	1.9513	2.5980e-06	2.9999

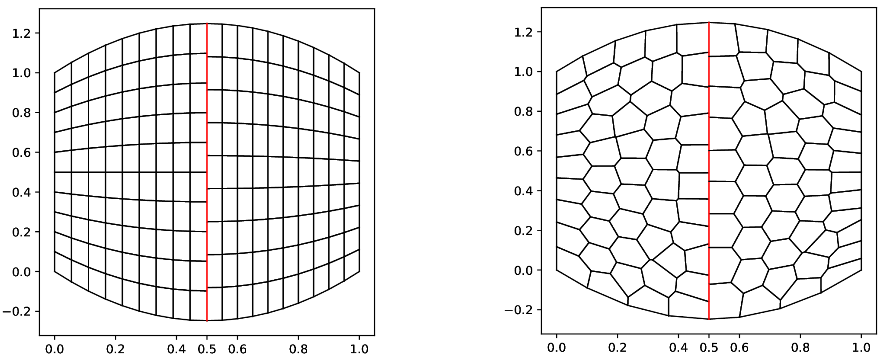

Fig. 5.4. $\mathtt{MESH}_7$ (left) and $\mathtt{MESH}_8$ (right) used in Example 5.4.

6. Conclusions and Discussions

The classical Nitsche-based projection method is incorporated into the framework of second-order SVEM to numerically solve a class of semilinear pseudo-parabolic equations on curved domains. Based on the characteristics of second-order SVEM, using the interpolation operator to approximate nonlinear terms can simplify the implementation. The optimal error estimates with respect to L^2-norm and an energy norm are proven.

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