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OPTIMAL ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD WITH GENERALIZED NUMERICAL FLUXES FOR ONE-DIMENSIONAL KDV TYPE EQUATIONS*

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Abstract

In this paper, we investigate the local discontinuous Galerkin method with generalized numerical fluxes for one-dimensional nonlinear Korteweg-de Vries type equations. The numerical flux for the nonlinear convection term is chosen as the generalized Lax-Friedrichs flux, and the generalized alternating flux and upwind-biased flux are used for the dispersion term. The generalized Lax-Friedrichs flux with anti-dissipation property will compensate the numerical dissipation of the dispersion term, resulting in a nearly energy conservative scheme that is useful in resolving waves and is beneficial for long time simulations. To deal with the nonlinearity and different numerical flux weights, a suitable numerical initial condition is constructed, for which a modified global projection is designed. By establishing relationships between the prime variable and auxiliary variables in combination with sharp bounds for jump terms, optimal error estimates are obtained. Numerical experiments are shown to confirm the validity of theoretical results.

Mathematics subject classification: 65M12, 65M15, 65M60.
Key words: Korteweg-de Vries type equations, Local discontinuous Galerkin method, Generalized fluxes, Error estimates.

1. Introduction

In this paper, we study the local discontinuous Galerkin (LDG) method with generalized numerical fluxes for one-dimensional nonlinear Korteweg-de Vries (KdV) type equations

$$u_t + f(u)_x + u_{xxx} = 0, \quad (x,t) \in I \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \qquad x \in I, \tag{1.1b}$$

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where $u_0(x)$ is a smooth function and $I = [a, b]$. The nonlinear function $f(u)$ is assumed to be sufficiently smooth with respect to u, and the exact solution u is smooth. The periodic boundary conditions are mainly considered, and the case with mixed boundary conditions is numerically investigated. For KdV equations, compared with the standard upwind and alternating fluxes, the energy conserving scheme will produce a lower growth of errors and is efficient in resolving waves. This can be achieved by choosing central fluxes for generalized KdV equations [1] or the generalized numerical fluxes with different weights for linearized KdV equations [11]. For nonlinear KdV type equations (1.1), by constructing a suitable numerical initial condition and a modified projection in combination with the relationships between the prime variable and auxiliary variables, optimal error estimates are derived.

The nonlinear KdV type equation is an important model for many nonlinear phenomena, which can describe wave phenomena in bubble-liquid mixtures [19], plasma physics [8] and anharmonic crystals [24]. There have been a variety of work on the theoretical and numerical aspects of KdV equations. For example, in [7], a generalized tanh function method was implemented to find the exact solutions of the KdV equation and the coupled KdV equation. A meshless method of lines was presented for the numerical solution of the KdV equation in [16]. Numerical solution of the KdV equation was obtained using the space-splitting technique and the differential quadrature method with cosine expansion [15].

The LDG method is an extension of the discontinuous Galerkin (DG) method. The DG method is a class of finite element method using discontinuous piecewise polynomials as the numerical solution and test functions, leading to advantages in high order accuracy, high parallel efficiency, flexibility for *hp*-adaptivity. It was first introduced to solve a linear steady-state hyperbolic equation [14] and was developed for solving nonlinear time dependent conservation laws [3,5]. The LDG method was proposed by Cockburn and Shu [4] to solve convection-diffusion equations. The main idea of the LDG method is to rewrite the original partial differential equation (PDE) involving high order spatial derivatives into an equivalent first order system and then the DG method can be applied. Later, it was actively applied to solve various high order equations. We refer to review papers [17, 21] for more details.

The LDG scheme for KdV type equations was first proposed in [23], in which stability property was shown for nonlinear case and suboptimal $(k + 1/2)$-th order was derived for the linear case. In [10], the method was extended to solve the nonlinear dispersive PDE involving compactly supported traveling wave solutions. For the LDG scheme solving nonlinear KdV equations, suboptimal $(k + 1/2)$-th order error estimate was obtained [20], and the loss of half an order is mainly due to some extra boundary terms arising from high order derivatives. By establishing several energy equations, optimal error estimate of order $k + 1$ is derived for linearized KdV equations [22]. Note that purely upwind and alternating fluxes are used in above work. For generalized KdV equations, a posteriori error estimates of conservative LDG methods is given [9]. In [25], for KdV type systems, four conservative and dissipative LDG schemes are proposed, in which the conservative/dissipative numerical fluxes are designed for the linear dispersion term and the nonlinear convection term, respectively. By virtue of some local Gauss-Radau projections, suboptimal error estimates of order $k+1/2$ are derived for dissipative fluxes, and numerical examples indicate that the conservative scheme performs better than the dissipative one for long time simulations.

In addition to the stability issue of the numerical fluxes in the design of scheme, the numerical viscosity plays an important role in resolving waves and for long time simulations. The LDG method with central and generalized alternating fluxes for solving the Burgers-Poisson equation was presented in [13]. The LDG scheme with upwind-biased and generalized alternating fluxes for linear convection-diffusion problem was discussed in [2]. In [11], the LDG method using generalized numerical fluxes for linearized KdV equations was studied and the optimal error estimate was obtained. For scalar nonlinear hyperbolic conservation laws, the generalized local Lax-Friedrichs (GLLF) flux that may not be monotone was proposed and optimal error estimate was shown in [12]. In these studies, according to different choices of numerical fluxes, generalized Gauss-Radau (GGR) projections were proposed, and an analysis of inverse of the coefficient matrix was essential. In particular, for linear PDEs with high order derivatives, a sub-family of numerical fluxes containing average values and jumps of numerical solutions were constructed in [6], which were proved to be of optimal $(k+1)$-th order by using some special projections. For the DG scheme with generalized fluxes solving wave equations, instead of analyzing the inverse of some matrices, [18] proposed an energy argument based on the coercivity of DG discretization operators and derived optimal error estimates, which can be easily extended to unstructured meshes.

In current work, we aim at analyzing the LDG method with generalized numerical fluxes for nonlinear KdV type equations. For initial error estimates, to deal with the nonlinearity of the equation, a corresponding nonlinear steady-state problem with a small enough constant is designed. As to the optimal error estimates of the LDG scheme, the main difficulty lies in the estimates of the boundary terms arising from the nonlinear term and different weights of generalized numerical fluxes. To do that, we first construct a modified GGR projection to eliminate the boundary terms of projection errors. Then, by establishing relationships between the prime variable and auxiliary variables in combination with sharp bounds for jump terms, optimal error estimates are derived. This provides a solid theoretical foundation for using generalized fluxes to solve KdV type equations and numerical experiments confirm the benefits of adjustable numerical viscosities for long time simulations, when compared to the standard upwind and alternating fluxes, see Figs. 5.2 and 5.3 below.

The rest of this paper is organized as follows. Section 2 is devoted to notation and the LDG scheme with generalized fluxes for nonlinear KdV type equations. Also, a numerical initial condition pertaining to a nonlinear steady-state problem is carefully designed, which implies optimal initial error estimates for auxiliary variables, prime variable and its time derivative solving (1.1). In Section 3, we begin by introducing GGR projections and defining a modified global projection. Then, we show the optimal initial error estimates. By establishing the relationships between the prime variable and auxiliary variables together with the control for jump terms, the optimal error estimate is derived in Section 4. In Section 5, numerical experiments are provided to verify theoretical results. Concluding remarks are given in Section 6.

2. The LDG Method

2.1. Notation

Let $\mathcal{I}_h = \{I_j = (x_{j-1/2}, x_{j+1/2})\}_{j=1}^N$ be a partition of $I = [a, b]$. The length of each element is $h_j = x_{j+1/2} - x_{j-1/2}$. The maximum element length is denoted by $h = \max_{1\le j\le N} h_j$. We assume the partition is quasi-uniform. That is, there exists a positive constant ν such that $h_j \ge \nu h$ for any j, as h goes to zero. The discontinuous finite element space is defined as

$$V_h^k = \big\{v \in L^2(I) : v|_{I_j} \in P^k(I_j), j = 1, \ldots, N\big\},$$

where $P^k(I_j)$ denotes the space of polynomials of degree at most k in I_j. We use

$$[\![p]\!]_{j+\frac{1}{2}} = p^+_{j+\frac{1}{2}} - p^-_{j+\frac{1}{2}}, \quad \{\!\{p\}\!\}_{j+\frac{1}{2}} = \frac{1}{2}(p^+_{j+\frac{1}{2}} + p^-_{j+\frac{1}{2}})$$

to represent the jump and the mean value of p at the element boundary. Furthermore, the weighted average is denoted as

$$p^{(\theta)}_{j+\frac{1}{2}} = \theta p^-_{j+\frac{1}{2}} + \widetilde{\theta} p^+_{j+\frac{1}{2}}, \quad \widetilde{\theta} = 1 - \theta.$$

Let $W^{\ell,p}(D)$ be the classical Sobolev space equipped with norm $\|\cdot\|_{\ell,p,D}$ for functions in D. In particular, the L^2-norm is denoted by $\|u\|^2_D = \int_D |u|^2 \mathrm{d}x$ and the L^∞-norm is denoted by $\|u\|_{\infty,D} = \max_{x\in D} |u(x)|$. The subscripts D, p, ℓ will be omitted when $D = I, p = 2$ or $\ell = 0$, and denote $W^{\ell,2}(D) = H^\ell(D)$. The broken Sobolev space $W^{\ell,p}(\mathcal{I}_h)$ and the corresponding norms can be defined in an analogous way. For example, the Sobolev norm is denoted as

$$\|u\|_{\ell,2,\mathcal{I}_h} \triangleq \|u\|_\ell = \left(\sum_{j=1}^N \|u\|^2_{\ell,I_j} \right)^{\frac{1}{2}}.$$

We use

$$\|u\|_{\Gamma_h} = \left(\sum_{j=1}^N \left(\left(u^-_{j+\frac{1}{2}}\right)^2 + \left(u^+_{j+\frac{1}{2}}\right)^2 \right) \right)^{\frac{1}{2}}$$

to denote the L^2-norm at cell boundaries.

2.2. Preliminaries

2.2.1. DG discretization operators

For notational convenience, we would like to denote the DG spatial discretization operators by

$$\mathcal{H}^\theta_j(v, w) = \int_{I_j} v w_x \mathrm{d}x - v^{(\theta)}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + v^{(\theta)}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}, \tag{2.1a}$$

$$\mathcal{H}^\wedge_j(v, w) = \int_{I_j} v w_x \mathrm{d}x - \hat{v}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}, \tag{2.1b}$$

and the removal of the subscript j indicates the summation of all j. Note that the operator (2.1a) is introduced for linear dispersive term with weighted numerical fluxes and the operator (2.1b) is defined for nonlinear convection term with the generalized Lax-Friedrichs (GLF) flux. The following properties of DG operators have been shown in [11].

Lemma 2.1. *Under the periodic boundary conditions, for $v, w \in H^1(\mathcal{I}_h)$, there holds*

$$\mathcal{H}^{\theta_1}(v, w) + \mathcal{H}^{\theta_2}(w, v) = (\widetilde{\theta}_2 - \theta_1) \sum_{j=1}^N ([\![v]\!][\![w]\!])_{j+\frac{1}{2}}, \tag{2.2a}$$

$$\mathcal{H}^\theta(w, w) = \left(\frac{1}{2} - \theta\right) \sum_{j=1}^N [\![w]\!]^2_{j+\frac{1}{2}}. \tag{2.2b}$$

2.2.2. Inverse inequalities

For $v_h \in V_h^k$, there exists a positive constant C independent of v_h and h such that

$$\|\partial_x v_h\| \le Ch^{-1}\|v_h\|, \tag{2.3}$$

$$\|v_h\|_{\Gamma_h} \le Ch^{-\frac{1}{2}}\|v_h\|, \tag{2.4}$$

$$\|v_h\|_\infty \le Ch^{-\frac{1}{2}}\|v_h\|. \tag{2.5}$$

2.3. The LDG scheme

As usual, we introduce two auxiliary variables q, p to rewrite (1.1) as

$$u_t + f(u)_x + p_x = 0, \quad p = q_x, \quad q = u_x. \tag{2.6}$$

Then, the semi-discrete LDG scheme is: $\forall\, t \in (0, T]$, to find

$$\big(u_h(t), q_h(t), p_h(t)\big) \in V_h^k \times V_h^k \times V_h^k \triangleq [V_h^k]^3$$

such that

$$\int_{I_j} u_{ht} v_h \mathrm{d}x = \mathcal{H}_j^{\wedge}\big(f(u_h), v_h\big) + \mathcal{H}_j^{\widetilde{\gamma}}(p_h, v_h), \tag{2.7a}$$

$$\int_{I_j} p_h w_h \mathrm{d}x = -\mathcal{H}_j^{\widetilde{\mu}}(q_h, w_h), \tag{2.7b}$$

$$\int_{I_j} q_h z_h \mathrm{d}x = -\mathcal{H}_j^{\gamma}(u_h, z_h) \tag{2.7c}$$

hold for any $(v_h, w_h, z_h) \in [V_h^k]^3$ and $j = 1, \ldots, N$. To possess adjustable numerical viscosities, we choose the generalized alternating and upwind-biased fluxes for the dispersion term

$$(\hat{u}_h)_{j+\frac{1}{2}} = (u_h)_{j+\frac{1}{2}}^{(\gamma)}, \quad (\hat{q}_h)_{j+\frac{1}{2}} = (q_h)_{j+\frac{1}{2}}^{(\widetilde{\mu})}, \quad (\hat{p}_h)_{j+\frac{1}{2}} = (p_h)_{j+\frac{1}{2}}^{(\widetilde{\gamma})} \tag{2.8a}$$

with $\gamma \neq \ 1/2$, $\mu > 1/2$, and the GLF flux analogous to [12] for the nonlinear convection term, namely

$$\hat{f}(u_h^-, u_h^+) = \left(\frac{1}{2} + \theta\right) f(u_h^-) + \left(\frac{1}{2} - \theta\right) f(u_h^+) - \lambda\alpha[\![u_h]\!], \tag{2.8b}$$

in which the subscript $j + 1/2$ is omitted, and $\alpha = \max_{u\in[m,M]} |f'(u)|$ with $[m, M]$ being the range of $u_0(x)$. We would like to emphasize that the GLF flux with a larger numerical viscosity coefficient is beneficial to the estimate of W_3 in the proof of Theorem 4.1 below.

2.4. The numerical initial condition

The numerical initial condition is chosen as the LDG approximation with fluxes (2.8) to a nonlinear steady-state problem

$$u + \tilde{\epsilon}_0 f(u)_x + \tilde{\epsilon}_0 u_{xxx} = u_0(x) + \tilde{\epsilon}_0 f\big(u_0(x)\big)_x + \tilde{\epsilon}_0 (u_0)_{xxx}(x) \tag{2.9}$$

equipped with periodic boundary conditions, where $\tilde{\epsilon}_0 > 0$ is a constant that will be specified in the derivation of (3.13), and $u_0(x) = u(x, 0)$ is the initial condition of (1.1). Optimal initial error estimates for auxiliary variables, the prime variable and its time derivative will be shown in Section 3.2. It is worth noting that $\tilde{\epsilon}_0$ is introduced to deal with the nonlinearity of the problem when generalized numerical fluxes are used, and $\tilde{\epsilon}_0 = 1$ for linearized KdV equations [11]. Moreover, a relationship between the steady-state problem (2.9) and the time dependent problem (1.1) is established in (3.19) below, which is essential to derive the optimal initial error estimate for time derivative of the error.

3. A New Projection and Initial Error Estimates

3.1. A new projection

Let us first recall the definition of the standard GGR projection [2]. For $z \in H^1(\mathcal{I}_h)$, the GGR projection $\mathbb{P}_h^{\psi} z$ ($\psi \neq 1/2$) is a global projection of z satisfying

$$\int_{I_j} (z - \mathbb{P}_h^{\psi} z) v_h \mathrm{d}x = 0, \quad \forall\, v_h \in P^{k-1}(I_j), \tag{3.1a}$$

$$\big(\mathbb{P}_h^{\psi} z\big)_{j-\frac{1}{2}}^{(\psi)} = z_{j-\frac{1}{2}}^{(\psi)}, \qquad j = 1, \ldots, N. \tag{3.1b}$$

For u, q, we can simply take the GGR projections $\mathbb{P}_h^{\gamma} u$ and $\mathbb{P}_h^{\widetilde{\mu}} q$, respectively.

Since the nonlinear convection term may change its flow direction, the projection errors of the prime variable u for the nonlinear convection and dispersion terms cannot be simultaneously eliminated by the projection $\mathbb{P}_h^{\gamma} u$. For this reason, we need to define a modified projection $\mathsf{P}_d^{\widetilde{\gamma}} p$ for p as

$$\int_{I_j} (p - \mathsf{P}_d^{\widetilde{\gamma}} p) v_h \mathrm{d}x = 0, \qquad \forall\, v_h \in P^{k-1}(I_j), \tag{3.2a}$$

$$\big(\mathsf{P}_d^{\widetilde{\gamma}}\big)_{j-\frac{1}{2}}^{\widetilde{\gamma}} = p_{j-\frac{1}{2}}^{(\widetilde{\gamma})} + d\,\widehat{f'(u)\big(u - \mathbb{P}_h^{\gamma} u\big)}_{j-\frac{1}{2}}, \quad j = 1, \ldots, N. \tag{3.2b}$$

Here, u is the exact solution of the Eq. (1.1), d is a constant, which is taken as $d = \epsilon_0^2$ with $\epsilon_0^3 = \tilde{\epsilon}_0$ for the estimate of (3.12d) in Lemma 3.2, and $d = 1$ for the proof of Theorem 4.1; that is, $\mathsf{P}_{\epsilon_0^2}^{\widetilde{\gamma}} \tilde{p}$ and $\mathsf{P}_1^{\widetilde{\gamma}} p$ will be used respectively. The term $\widehat{f'(u)(u - \mathbb{P}_h^{\gamma} u)}$ depends on the choice of flux $\hat{f}(u_h^-, u_h^+)$, and its specific expression, namely

$$\widehat{f'(u)(u - \mathbb{P}_h^{\gamma} u)} = \left(\frac{1}{2} + \theta\right) f'(u)\big(u - \mathbb{P}_h^{\gamma} u\big)^- + \left(\frac{1}{2} - \theta\right) f'(u)\big(u - \mathbb{P}_h^{\gamma} u\big)^+ - \lambda\alpha [\![u - \mathbb{P}_h^{\gamma} u]\!],$$

or

$$\widehat{f'(u)\big(u - \mathbb{P}_h^{\gamma} u\big)} = f'(u)\{\!\{ u - \mathbb{P}_h^{\gamma} u \}\!\} - (\theta f'(u) + \lambda\alpha) [\![u - \mathbb{P}_h^{\gamma} u]\!]$$

can be obtained by the definition of GLF flux and the Taylor expansion, where α is the same as that in (2.8b), and the subscript $j + 1/2$ is omitted. It can be seen that $\mathsf{P}_d^{\widetilde{\gamma}} p$ depends on $\mathbb{P}_h^{\gamma} u$ (already known), making $\mathbb{P}_h^{\gamma} u$ and $\mathsf{P}_d^{\widetilde{\gamma}} p$ become a pair of projections. The new projection $\mathsf{P}_d^{\widetilde{\gamma}} p$ is beneficial to balance the projection errors of $p - \mathsf{P}_d^{\widetilde{\gamma}} p$ and $df'(u)(u - \mathbb{P}_h^{\gamma} u)$, see (3.12d) and the estimate of Π_4 in (4.9a) below.

The GGR projection $\mathbb{P}_h^{\psi}$ defined in (3.1) exists and is unique, and for $z \in H^{k+1}(\mathcal{I}_h)$, there holds the optimal approximation property [2]

$$\big\|z - \mathbb{P}_h^{\psi} z\big\| + h\big\|z - \mathbb{P}_h^{\psi} z\big\|_{\infty} + h^{\frac{1}{2}}\big\|z - \mathbb{P}_h^{\psi} z\big\|_{\Gamma_h} \le C h^{k+1} \|z\|_{k+1}. \tag{3.3}$$

For the modified projection $\mathsf{P}_d^{\widetilde{\gamma}}$, the optimal projection property can be proved. Moreover, by (3.2b), $\mathsf{P}_d^{\widetilde{\gamma}}$ depends on $f'(u)$ and thus t, indicating that $(\mathsf{P}_d^{\widetilde{\gamma}} p)_t \neq \mathsf{P}_d^{\widetilde{\gamma}}(p_t)$ and the optimal approximation property of the time derivative of the projection error is not trivial. Fortunately, due to the fact that $(\mathbb{P}_h^{\gamma} u)_t = \mathbb{P}_h^{\gamma}(u_t)$, the optimal approximation property of the time derivative can be shown, which is given in the following lemma.

Lemma 3.1. *Under periodic boundary conditions, the modified projection $\mathsf{P}_d^{\widetilde{\gamma}} w$ exists and is unique. Moreover, assuming $\partial_t^m w \in H^{k+1}(\mathcal{I}_h)$ and the exact solution u of (1.1) satisfies $\partial_t^m u \in H^{k+1}(\mathcal{I}_h)$, we have the following optimal approximation property:*

$$\|\partial_t^m (w - \mathsf{P}_d^{\widetilde{\gamma}} w)\| \le C h^{k+1} \big(\|\partial_t^m u\|_{k+1} + \|\partial_t^m w\|_{k+1} \big),$$

where $m = 0, 1, 2$ and C is independent of h.

Proof. We will show the property of $\mathsf{P}_d^{\widetilde{\gamma}} w$ by the property of GGR projection $\mathbb{P}_h^{\widetilde{\gamma}} w$. To do that, let $W = \mathsf{P}_d^{\widetilde{\gamma}} w - \mathbb{P}_h^{\widetilde{\gamma}} w$. Because $\mathbb{P}_h^{\widetilde{\gamma}} w$ exists and is unique, we can obtain existence and uniqueness of the projection $\mathsf{P}_d^{\widetilde{\gamma}} w$ if W exists and is unique. Denote by $W_j(x)$ the restriction of W to each element I_j

$$W_j(x) = \sum_{\ell=0}^{k} \alpha_{j,\ell} P_{j,\ell}(x) = \sum_{\ell=0}^{k} \alpha_{j,\ell} P_\ell(s),$$

where $P_\ell(s)$ is the ℓ-th order Legendre polynomials in $[-1, 1]$ with $s = 2(x - x_j)/h_j$, and $x_j = (x_{j+1/2} + x_{j-1/2})/2$ is the center of I_j. By the definitions of $\mathbb{P}_h^{\widetilde{\gamma}} w, \mathsf{P}_d^{\widetilde{\gamma}} w$, we have

$$\int_{I_j} W v_h \mathrm{d}x = 0, \qquad \forall\, v_h \in P^{k-1}(I_j), \tag{3.4a}$$

$$W_{j-\frac{1}{2}}^{(\widetilde{\gamma})} = d\widehat{f'(u)(u - \mathbb{P}_h^{\gamma} u)}_{j-\frac{1}{2}}, \quad j = 1, \ldots, N. \tag{3.4b}$$

From equality (3.4a) and the orthogonality of $P_\ell(s)$, we obtain

$$\alpha_{j,\ell} = 0, \quad \ell = 0, \ldots, k-1, \quad j = 1, \ldots, N.$$

Thus, $W_j(x) = \alpha_{j,k} P_k(s)$, and $\alpha_{j,k}$ is determined by (3.4b). Substituting the values of Legendre polynomials at the endpoints into (3.4b), we arrive at

$$\widetilde{\gamma} \alpha_{j-1,k} + (-1)^k \gamma \alpha_{j,k} = d\widehat{f'(u)\big(u - \mathbb{P}_h^{\gamma} u\big)}_{j-\frac{1}{2}} \triangleq c_j, \quad j = 1, \ldots, N,$$

which, for periodic boundary conditions, is the following linear system:

$$\boldsymbol{A} \boldsymbol{\alpha}_k = \boldsymbol{c},$$

where $\boldsymbol{A} = \mathrm{circ}((-1)^k \gamma, 0, \cdots, 0, \widetilde{\gamma})$ is an $N \times N$ circulant matrix, and $\boldsymbol{\alpha}_k = [\alpha_{1,k}, \cdots, \alpha_{N,k}]^{\mathrm{T}}$, $\boldsymbol{c} = [c_1, \cdots, c_N]^{\mathrm{T}}$. The determinant of $\boldsymbol{A}$ is

$$|\boldsymbol{A}| = \big((-1)^k \gamma\big)^N - (-\widetilde{\gamma})^N.$$

Clearly, the determinant of $\boldsymbol{A}$ is always not equal to 0 when $\gamma \neq 1/2$. Therefore, the matrix $\boldsymbol{A}$ is invertible. This implies the existence and uniqueness of W, and thus $\mathsf{P}_d^{\widetilde{\gamma}} w$.

In what follows, we first prove the approximation property $\|w - \mathsf{P}_d^{\widetilde{\gamma}} w\| \le C h^{k+1}$. Since the inverse of a nonsingular circulant matrix is also circulant, we have

$$\boldsymbol{A}^{-1} = \frac{1}{(-1)^k \gamma \big(1 - q^N\big)} \mathrm{circ}(1, q, \cdots, q^{N-1}),$$

where $q = -\widetilde{\gamma}/(\gamma(-1)^k)$ ($|q| \neq 1$). The row- and column-norm of the $\boldsymbol{A}^{-1}$ are equal and satisfy

$$\|\boldsymbol{A}^{-1}\|_1 = \|\boldsymbol{A}^{-1}\|_\infty \leq \frac{1}{\big|\gamma(1-|q|)\big|},$$

hence the spectral norm satisfies

$$\|\boldsymbol{A}^{-1}\|^2 \leq \|\boldsymbol{A}^{-1}\|_1 \|\boldsymbol{A}^{-1}\|_\infty \leq \frac{1}{\gamma^2(1-|q|)^2}.$$

Consequently,

$$\|\boldsymbol{\alpha}_k\|^2 = \|\boldsymbol{A}^{-1}\boldsymbol{c}\|^2 \leq \|\boldsymbol{A}^{-1}\|^2 \|\boldsymbol{c}\|^2 \leq C\|\boldsymbol{c}\|^2.$$

To estimate $\|\boldsymbol{c}\|^2$, let us consider the bound for $c_j = d\widehat{f'(u)(u - \mathbb{P}_h^\gamma u)}_{j-1/2}$. Since

$$\widehat{f'(u)\big(u - \mathbb{P}_h^\gamma ub\big)} = f'(u)\big\{\!\!\big\{u - \mathbb{P}_h^\gamma u\big\}\!\!\big\} - \big(\theta f'(u) + \lambda\alpha\big)\big[\!\!\big[u - \mathbb{P}_h^\gamma u\big]\!\!\big], \tag{3.5}$$

where α is the same as that in (2.8b), one has

$$|c_j|^2 \leq C\Big(\big[\!\!\big[u - \mathbb{P}_h^\gamma u\big]\!\!\big]_{j-\frac{1}{2}}^2 + \big\{\!\!\big\{u - \mathbb{P}_h^\gamma u\big\}\!\!\big\}_{j-\frac{1}{2}}^2\Big),$$

which, by (3.3), implies

$$\|\boldsymbol{c}\|^2 \leq C\big\|u - \mathbb{P}_h^\gamma u\big\|_{\Gamma_h}^2 \leq Ch^{2k+1}\|u\|_{k+1}^2,$$

where the positive constant C is independent of h. Therefore,

$$\|\boldsymbol{\alpha}_k\|^2 \leq Ch^{2k+1}\|u\|_{k+1}^2.$$

This, together with the fact that

$$\|W\|^2 = \sum_{j=1}^N \alpha_{j,k}^2 \|P_{j,k}(x)\|_{I_j}^2 = \sum_{j=1}^N \frac{h_j \alpha_{j,k}^2}{2k+1} \leq Ch\|\boldsymbol{\alpha}_k\|^2,$$

produces the approximation result of $\|W\|$ and thus

$$\|w - \mathsf{P}_d^{\widetilde{\gamma}} w\| \leq Ch^{k+1}\big(\|u\|_{k+1} + \|w\|_{k+1}\big).$$

Next, we turn to the proof of $\|\partial_t(w - \mathsf{P}_d^{\widetilde{\gamma}} w)\| \leq Ch^{k+1}$. Let $W_t = (\mathsf{P}_d^{\widetilde{\gamma}} w)_t - (\mathbb{P}_h^{\widetilde{\gamma}} w)_t$ and denote its restriction to each element I_j as

$$(W_t)_j(x) = \sum_{\ell=0}^k \beta_{j,\ell} P_{j,\ell}(x) = \sum_{\ell=0}^k \beta_{j,\ell} P_\ell(s).$$

Taking the time derivative of (3.4), we obtain

$$\int_{I_j} W_t v_h \mathrm{d}x = 0, \qquad \forall\, v_h \in P^{k-1}(I_j), \tag{3.6a}$$

$$(W_t)_{j-\frac{1}{2}}^{(\widetilde{\gamma})} = d\big(\widehat{f'(u)\big(u - \mathbb{P}_h^\gamma u\big)}\big)_t\big|_{j-\frac{1}{2}}, \quad j = 1, \ldots, N. \tag{3.6b}$$

By (3.6a), we have

$$\beta_{j,\ell} = 0, \quad \ell = 0, \ldots, k-1, \quad j = 1, \ldots, N,$$

which implies $(W_t)_j(x) = \beta_{j,k} P_k(s)$, and further by (3.6b)

$$\widetilde{\gamma}\beta_{j-1,k} + (-1)^k \gamma \beta_{j,k} = d\big(f'(\widehat{u)\big(u - \mathbb{P}_h^{\gamma} u}\big)\big)_t\Big|_{j-\frac{1}{2}} \triangleq (c_t)_j, \quad j = 1, \ldots, N.$$

Thus,

$$\boldsymbol{A}\boldsymbol{\beta}_k = \boldsymbol{c}_t,$$

where $\boldsymbol{A} = \text{circ}((-1)^k\gamma, 0, \cdots, 0, \widetilde{\gamma})$ is an $N \times N$ circulant matrix, and $\boldsymbol{\beta}_k = [\beta_{1,k}, \cdots, \beta_{N,k}]^{\mathrm{T}}$, $\boldsymbol{c}_t = [(c_t)_1, \cdots, (c_t)_N]^{\mathrm{T}}$. By the above estimate

$$\|\boldsymbol{A}^{-1}\|^2 \le \frac{1}{\gamma^2(1-|q|)^2},$$

we have

$$\|\boldsymbol{\beta}_k\|^2 = \big\|\boldsymbol{A}^{-1}\boldsymbol{c}_t\big\|^2 \le \|\boldsymbol{A}^{-1}\|^2 \|\boldsymbol{c}_t\|^2 \le C\|\boldsymbol{c}_t\|^2.$$

Moreover, using the estimate of $\|\boldsymbol{c}\|^2$ and taking into account $(\mathbb{P}_h^{\gamma} u)_t = \mathbb{P}_h^{\gamma}(u_t)$, it is easy to show

$$\|\boldsymbol{c}_t\|^2 \le C\big\|(u - \mathbb{P}_h^{\gamma} u)_t\big\|_{\Gamma_h}^2 \le Ch^{2k+1}\|u_t\|_{k+1}^2,$$

where the positive constant C is independent of h. Therefore,

$$\|\boldsymbol{\beta}_k\|^2 \le Ch^{2k+1}\|u_t\|_{k+1}^2.$$

Combining the fact that

$$\|W_t\|^2 = \sum_{j=1}^{N} \beta_{j,k}^2 \|P_{j,k}(x)\|_{I_j}^2 = \sum_{j=1}^{N} \frac{h_j \beta_{j,k}^2}{2k+1} \le Ch\|\boldsymbol{\beta}_k\|^2,$$

we can obtain

$$\|W_t\| \le Ch^{k+1}\|u_t\|_{k+1},$$

and thus

$$\big\|\partial_t(w - \mathsf{P}_d^{\widetilde{\gamma}} w)\big\| \le Ch^{k+1}\big(\|u_t\|_{k+1} + \|w_t\|_{k+1}\big).$$

Finally,

$$\big\|\partial_{tt}(w - \mathsf{P}_d^{\widetilde{\gamma}} w)\big\| \le Ch^{k+1}\big(\|u_{tt}\|_{k+1} + \|w_{tt}\|_{k+1}\big)$$

can be obtained in an analogous way. This completes the proof of Lemma 3.1. □

By above GGR projections, we are now ready to split e_u, e_q, e_p into

$$\begin{aligned}
e_u &= u - u_h = \big(\mathbb{P}_h^{\gamma} u - u_h\big) + \big(u - \mathbb{P}_h^{\gamma} u\big) \triangleq \xi_u + \eta_u, \\
e_q &= q - q_h = \big(\mathbb{P}_h^{\widetilde{\mu}} q - q_h\big) + \big(q - \mathbb{P}_h^{\widetilde{\mu}} q\big) \triangleq \xi_q + \eta_q, \\
e_p &= p - p_h = \big(\mathsf{P}_d^{\widetilde{\gamma}} p - p_h\big) + \big(p - \mathsf{P}_d^{\widetilde{\gamma}} p\big) \triangleq \xi_p + \eta_p,
\end{aligned}$$

where $\mathsf{P}_{\epsilon_0^2}^{\widetilde{\gamma}}$ and $\mathsf{P}_1^{\widetilde{\gamma}}$ will be used for the estimates when $t = 0$ and $t > 0$, respectively. Obviously, by the definitions of $\mathbb{P}_h^{\gamma}$ and $\mathbb{P}_h^{\widetilde{\mu}}$, one has

$$\mathcal{H}^{\gamma}(\eta_u, v_h) = 0, \quad \mathcal{H}^{\widetilde{\mu}}(\eta_q, v_h) = 0, \quad \forall\, v_h \in V_h^k. \tag{3.7}$$

3.2. Initial error estimates

For nonlinear KdV type equations (1.1), the nonlinear term, especially S_3-S_5 in (3.11), will bring extra trouble for initial error estimates if we take the same numerical initial condition as that for the linearized case [11]. Therefore, a small positive constant $\tilde{\epsilon}_0$ is introduced in (2.9), which is independent of h. Besides, there are additional differences compared with the proof for the linear case. For example, a modified projection $\mathsf{P}_{\epsilon_0^2}^{\widetilde{\gamma}}$ is needed, and some relationships of the LDG solutions as well as the jump terms are constructed.

Lemma 3.2. *For the numerical initial condition* (2.9) *with* $\tilde{\epsilon}_0 = 1/(24C_\star)$*, periodic boundary conditions and* $k \geq 1$*, assuming* $u_0 \in H^{k+2}(\mathcal{I}_h)$ *and* $\|u_0 - u_h(0)\|_\infty \leq Ch$*, we have the following optimal initial error estimates for time dependent nonlinear KdV type equations* (1.1)*:*

$$\|u_0 - u_h(0)\| + \|q_0 - q_h(0)\| + \|p_0 - p_h(0)\| + \|u_t(0) - u_{ht}(0)\| \leq Ch^{k+1},$$

where $q_0 = (u_0)_x, p_0 = (u_0)_{xx}$*, and* C *is independent of* h*.*

Proof. Clearly, the LDG scheme with fluxes (2.8) for the steady-state problem (2.9) is

$$\begin{aligned}
&\int_{I_j} \tilde{u}_h v_h \mathrm{d}x = \tilde{\epsilon}_0 \mathcal{H}_j^{\wedge}\big(f(\tilde{u}_h), v_h\big) + \epsilon_0 \mathcal{H}_j^{\widetilde{\gamma}}(\tilde{p}_h, v_h) + \int_{I_j} g(x) v_h \mathrm{d}x, \\
&\int_{I_j} \tilde{p}_h w_h \mathrm{d}x = -\epsilon_0 \mathcal{H}_j^{\widetilde{\mu}}(\tilde{q}_h, w_h), \\
&\int_{I_j} \tilde{q}_h z_h \mathrm{d}x = -\epsilon_0 \mathcal{H}_j^{\gamma}(\tilde{u}_h, z_h),
\end{aligned}$$

where the positive constant $\tilde{\epsilon}_0$ has been introduced in (2.9) with $\epsilon_0^3 = \tilde{\epsilon}_0$, and $\tilde{\epsilon}_0$ will be specified in the estimate of (3.13). Here and below,

$$g(x) = u_0(x) + \tilde{\epsilon}_0 f\big(u_0(x)\big)_x + \tilde{\epsilon}_0 (u_0)_{xxx}(x),$$

the exact solution and numerical solution of the steady-state problem are $\tilde{u}, \tilde{q}, \tilde{p}$ and $\tilde{u}_h, \tilde{q}_h, \tilde{p}_h$ respectively, and

$$\tilde{u} = u_0, \quad \tilde{q} = \epsilon_0 \tilde{u}_x = \epsilon_0 (u_0)_x = \epsilon_0 q_0, \quad \tilde{p} = \epsilon_0 \tilde{q}_x = \epsilon_0^2 (u_0)_{xx} = \epsilon_0^2 p_0.$$

By the continuity of numerical solutions with respect to time, the LDG scheme (2.7) for the time dependent problem still holds when $t = 0$. This implies that $\tilde{u}_h = u_h(0), \tilde{q}_h = \epsilon_0 q_h(0)$, $\tilde{p}_h = \epsilon_0^2 p_h(0)$. Thus, the following relationships hold:

$$\begin{aligned}
&e_{\tilde{u}} = e_u(0), \quad e_{\tilde{q}} = \epsilon_0 e_q(0), \quad e_{\tilde{p}} = \epsilon_0^2 e_p(0), \\
&\xi_{\tilde{u}} = \xi_u(0), \quad \xi_{\tilde{q}} = \epsilon_0 \xi_q(0), \quad \xi_{\tilde{p}} = \epsilon_0^2 \big(\mathsf{P}_{\epsilon_0^2}^{\widetilde{\gamma}} p_0 - p_h(0)\big), \\
&\eta_{\tilde{u}} = \eta_u(0), \quad \eta_{\tilde{q}} = \epsilon_0 \eta_q(0), \quad \eta_{\tilde{p}} = \epsilon_0^2 \big(p_0 - \mathsf{P}_{\epsilon_0^2}^{\widetilde{\gamma}} p_0\big).
\end{aligned} \tag{3.8}$$

The optimal initial error estimates can be obtained by establishing the estimates to $\|\xi_{\tilde{u}}\|, \|\xi_{\tilde{q}}\|$, $\|\xi_{\tilde{p}}\|$ and the jump terms, which are divided into the following five steps.

Step 1: Error equations. By Galerkin orthogonality, the error decomposition (3.8), the second order Taylor expansion

$$\begin{aligned}
f(\tilde{u}) - f(\tilde{u}_h) &= f'(\tilde{u}) e_{\tilde{u}} - e_{\tilde{u}}^2 \int_0^1 f''\big(\tilde{u} + s(\tilde{u}_h - \tilde{u})\big)(1-s) \mathrm{d}s \\
&\triangleq f'(\tilde{u}) \xi_{\tilde{u}} + f'(\tilde{u}) \eta_{\tilde{u}} - \tilde{R} e_{\tilde{u}}^2,
\end{aligned} \tag{3.9}$$

and summing over all elements, there hold the following error equations:

$$\int_I \xi_{\tilde{u}} v_h \mathrm{d}x = -\int_I \eta_{\tilde{u}} v_h \mathrm{d}x + \tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(f'(\tilde{u})\xi_{\tilde{u}}, v_h\big) + \tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(f'(\tilde{u})\eta_{\tilde{u}}, v_h\big)$$
$$- \tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(\tilde{R}e_{\tilde{u}}^2, v_h\big) + \epsilon_0 \mathcal{H}^{\widetilde{\gamma}}(\xi_{\tilde{p}}, v_h) + \epsilon_0 \mathcal{H}^{\widetilde{\gamma}}(\eta_{\tilde{p}}, v_h), \tag{3.10a}$$

$$\int_I \xi_{\tilde{p}} w_h \mathrm{d}x = -\int_I \eta_{\tilde{p}} w_h \mathrm{d}x - \epsilon_0 \mathcal{H}^{\widetilde{\mu}}(\xi_{\tilde{q}}, w_h) - \epsilon_0 \mathcal{H}^{\widetilde{\mu}}(\eta_{\tilde{q}}, w_h), \tag{3.10b}$$

$$\int_I \xi_{\tilde{q}} z_h \mathrm{d}x = -\int_I \eta_{\tilde{q}} z_h \mathrm{d}x - \epsilon_0 \mathcal{H}^{\gamma}(\xi_{\tilde{u}}, z_h) - \epsilon_0 \mathcal{H}^{\gamma}(\eta_{\tilde{u}}, z_h). \tag{3.10c}$$

Step 2: Estimate to $\|\xi_{\tilde{u}}\|$. Taking $(v_h, w_h, z_h) = (\xi_{\tilde{u}}, \xi_{\tilde{q}}, -\xi_{\tilde{p}})$ in (3.10), summing over all j and using (3.7), we have

$$\|\xi_{\tilde{u}}\|^2 = S_1 + S_2 + S_3 + S_4 + S_5, \tag{3.11}$$

where

$$\begin{aligned}
S_1 &= -\int_I \eta_{\tilde{u}} \xi_{\tilde{u}} \mathrm{d}x - \int_I \eta_{\tilde{p}} \xi_{\tilde{q}} \mathrm{d}x + \int_I \eta_{\tilde{q}} \xi_{\tilde{p}} \mathrm{d}x,\\
S_2 &= \epsilon_0\big(\mathcal{H}^{\widetilde{\gamma}}(\xi_{\tilde{p}}, \xi_{\tilde{u}}) + \mathcal{H}^{\gamma}(\xi_{\tilde{u}}, \xi_{\tilde{p}}) - \mathcal{H}^{\widetilde{\mu}}(\xi_{\tilde{q}}, \xi_{\tilde{q}})\big),\\
S_3 &= \tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{u}}\big),\\
S_4 &= \tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(f'(\tilde{u})\eta_{\tilde{u}}, \xi_{\tilde{u}}\big) + \epsilon_0 \mathcal{H}^{\widetilde{\gamma}}(\eta_{\tilde{p}}, \xi_{\tilde{u}}),\\
S_5 &= -\tilde{\epsilon}_0 \mathcal{H}^{\wedge}\big(\tilde{R}e_{\tilde{u}}^2, \xi_{\tilde{u}}\big),
\end{aligned}$$

which will be estimated separately. By the optimal approximation properties of projections and Young's inequality,

$$S_1 \le \frac{1}{8}\|\xi_{\tilde{u}}\|^2 + \epsilon_1\big(\|\xi_{\tilde{q}}\|^2 + \|\xi_{\tilde{p}}\|^2\big) + Ch^{2k+2}, \tag{3.12a}$$

where ϵ_1 can be small enough and C is a constant independent of h. S_2 can be estimated by Lemma 2.1 and $\mu > 1/2$; it reads,

$$S_2 = -\epsilon_0\left(\mu - \frac{1}{2}\right)\sum_{j=1}^{N} [\![\xi_{\tilde{q}}]\!]_{j+\frac{1}{2}}^2 \le 0. \tag{3.12b}$$

For S_3, it is easy to show that

$$\begin{aligned}
S_3 &= \tilde{\epsilon}_0\left(\sum_{j=1}^{N}\int_{I_j} f'(\tilde{u})\xi_{\tilde{u}}(\xi_{\tilde{u}})_x \mathrm{d}x + \sum_{j=1}^{N}\big(\widehat{f'(\tilde{u})\xi_{\tilde{u}}}\big)_{j+\frac{1}{2}}[\![\xi_{\tilde{u}}]\!]_{j+\frac{1}{2}}\right)\\
&= -\frac{\tilde{\epsilon}_0}{2}\sum_{j=1}^{N}\int_{I_j} f'(\tilde{u})_x \xi_{\tilde{u}}^2 \mathrm{d}x - \tilde{\epsilon}_0\sum_{j=1}^{N}\big(f'(\tilde{u})\{\!\{\xi_{\tilde{u}}\}\!\}[\![\xi_{\tilde{u}}]\!]\big)_{j+\frac{1}{2}}\\
&\quad + \tilde{\epsilon}_0\sum_{j=1}^{N}\big(f'(\tilde{u})\{\!\{\xi_{\tilde{u}}\}\!\}[\![\xi_{\tilde{u}}]\!] - \big(\theta f'(\tilde{u}) + \lambda\tilde{\alpha}\big)[\![\xi_{\tilde{u}}]\!]^2\big)_{j+\frac{1}{2}}\\
&= -\frac{\tilde{\epsilon}_0}{2}\sum_{j=1}^{N}\int_{I_j} f'(\tilde{u})_x \xi_{\tilde{u}}^2 \mathrm{d}x - \tilde{\epsilon}_0\sum_{j=1}^{N}\big(\theta f'(\tilde{u}) + \lambda\tilde{\alpha}\big)_{j+\frac{1}{2}}[\![\xi_{\tilde{u}}]\!]_{j+\frac{1}{2}}^2,
\end{aligned}$$

where we have also used the following expression of $\widehat{f'(\tilde{u})\xi_{\tilde{u}}}$ implied by the definition of GLF flux and Taylor expansion:

$$\begin{aligned}\widehat{f'(\tilde{u})\xi_{\tilde{u}}} &= \left(\frac{1}{2}+\theta\right) f'(\tilde{u})\xi_{\tilde{u}}^- + \left(\frac{1}{2}-\theta\right) f'(\tilde{u})\xi_{\tilde{u}}^+ - \lambda\tilde{\alpha}[\![\xi_{\tilde{u}}]\!] \\ &= f'(\tilde{u})\{\!\{\xi_{\tilde{u}}\}\!\} - \big(\theta f'(\tilde{u}) + \lambda\tilde{\alpha}\big)[\![\xi_{\tilde{u}}]\!],\end{aligned}$$

where $\tilde{\alpha} = \alpha(0)$ is the value of α at $t = 0$, and α is the same as that in (2.8b). By the definition of $\tilde{\alpha}$ and $\lambda \geq |\theta|$, we have

$$-\big(\theta f'(\tilde{u}) + \lambda\tilde{\alpha}\big)_{j+\frac{1}{2}} \leq 0.$$

Consequently,

$$S_3 \leq C_\star \tilde{\epsilon}_0 \|\xi_{\tilde{u}}\|^2. \tag{3.12c}$$

Here and below, $C_\star$ represents a constant depending on f''. Using the definitions of $\mathbb{P}_h^\gamma, \mathsf{P}_d^{\tilde{\gamma}}$ and the inverse inequality (2.3), we obtain

$$\begin{aligned} S_4 &= \tilde{\epsilon}_0\left(\sum_{j=1}^N \int_{I_j} f'(\tilde{u})\eta_{\tilde{u}}(\xi_{\tilde{u}})_x \mathrm{d}x + \sum_{j=1}^N \big(\widehat{f'(\tilde{u})\eta_{\tilde{u}}}\big)_{j+\frac{1}{2}}[\![\xi_{\tilde{u}}]\!]_{j+\frac{1}{2}}\right) \\ &\quad + \epsilon_0\left(\sum_{j=1}^N \int_{I_j} \eta_{\tilde{p}}(\xi_{\tilde{u}})_x \mathrm{d}x + \sum_{j=1}^N \big(\eta_{\tilde{p}}^{(\tilde{\gamma})}[\![\xi_{\tilde{u}}]\!]\big)_{j+\frac{1}{2}}\right) \\ &= \tilde{\epsilon}_0 \sum_{j=1}^N \left(\int_{I_j} f'(\tilde{u}_j)\eta_{\tilde{u}}(\xi_{\tilde{u}})_x \mathrm{d}x + \int_{I_j} \big(f'(\tilde{u}) - f'(\tilde{u}_j)\big)\eta_{\tilde{u}}(\xi_{\tilde{u}})_x \mathrm{d}x\right) \\ &\leq C_\star \tilde{\epsilon}_0 \|\xi_{\tilde{u}}\|^2 + Ch^{2k+2}. \end{aligned} \tag{3.12d}$$

For the high order term S_5, by the assumption $\|u_0 - u_h(0)\|_\infty \leq Ch$, the inverse inequalities and the approximation properties of projections, one has

$$\begin{aligned} S_5 &= -\tilde{\epsilon}_0\left(\sum_{j=1}^N \int_{I_j} \tilde{R}e_{\tilde{u}}^2(\xi_{\tilde{u}})_x \mathrm{d}x + \sum_{j=1}^N \widehat{(\tilde{R}e_{\tilde{u}}^2)}_{j+\frac{1}{2}}[\![\xi_{\tilde{u}}]\!]_{j+\frac{1}{2}}\right) \\ &\leq C_\star \tilde{\epsilon}_0 \|\xi_{\tilde{u}}\|^2 + Ch^{2k+2}. \end{aligned} \tag{3.12e}$$

Collecting (3.12a)-(3.12e) into (3.11) and taking $\tilde{\epsilon}_0 = 1/(24C_\star)$, we arrive at

$$\begin{aligned} \|\xi_{\tilde{u}}\|^2 &\leq \left(\frac{1}{8} + 3C_\star\tilde{\epsilon}_0\right)\|\xi_{\tilde{u}}\|^2 + \epsilon_1\big(\|\xi_{\tilde{q}}\|^2 + \|\xi_{\tilde{p}}\|^2\big) + Ch^{2k+2} \\ &\leq \frac{1}{4}\|\xi_{\tilde{u}}\|^2 + \epsilon_1\big(\|\xi_{\tilde{q}}\|^2 + \|\xi_{\tilde{p}}\|^2\big) + Ch^{2k+2}. \end{aligned} \tag{3.13}$$

Step 3: Estimate to $\|\xi_{\tilde{q}}\|$. Prior to estimating $\|\xi_{\tilde{q}}\|$, let us first consider the bounds for $\mathcal{H}^\wedge(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}}), \mathcal{H}^\wedge(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{p}})$ and the jump terms $[\![\xi_{\tilde{u}}]\!]^2, [\![\xi_{\tilde{q}}]\!]^2, [\![\xi_{\tilde{p}}]\!]^2$. To this end, we rewrite $\mathcal{H}^\wedge(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}})$ as

$$\mathcal{H}^\wedge\big(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}}\big) = \sum_{j=1}^N \int_{I_j} f'(\tilde{u})\xi_{\tilde{u}}(\xi_{\tilde{q}})_x \mathrm{d}x + \sum_{j=1}^N \big(\widehat{f'(\tilde{u})\xi_{\tilde{u}}}\big)_{j+\frac{1}{2}}[\![\xi_{\tilde{q}}]\!]_{j+\frac{1}{2}}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \int_{I_j} f'(\tilde{u})\xi_{\tilde{u}}(\xi_{\tilde{q}})_x \mathrm{d}x + \sum_{j=1}^{N} \left(f'(\tilde{u})\xi_{\tilde{u}}^{(\gamma)} [\![\xi_{\tilde{q}}]\!] \right)_{j+\frac{1}{2}} \\
&\quad + \sum_{j=1}^{N} \left(\left(\gamma - \theta - \frac{1}{2} \right) f'(\tilde{u}) - \lambda\tilde{\alpha} \right)_{j+\frac{1}{2}} \left([\![\xi_{\tilde{u}}]\!] [\![\xi_{\tilde{q}}]\!] \right)_{j+\frac{1}{2}} \\
&= \sum_{j=1}^{N} f'(\tilde{u}_{j+\frac{1}{2}}) \mathcal{H}_j^{\gamma}(\xi_{\tilde{u}}, \xi_{\tilde{q}}) + \sum_{j=1}^{N} \int_{I_j} (f'(\tilde{u}) - f'(\tilde{u}_{j+\frac{1}{2}}))\xi_{\tilde{u}}(\xi_{\tilde{q}})_x \mathrm{d}x \\
&\quad + \sum_{j=1}^{N} \left(\left(\gamma - \theta - \frac{1}{2} \right) f'(\tilde{u}) - \lambda\tilde{\alpha} \right)_{j+\frac{1}{2}} \left([\![\xi_{\tilde{u}}]\!] [\![\xi_{\tilde{q}}]\!] \right)_{j+\frac{1}{2}} \\
&\quad - \sum_{j=1}^{N} \left(f'(\tilde{u}_{j+\frac{3}{2}}) - f'(\tilde{u}_{j+\frac{1}{2}}) \right) \left(\xi_{\tilde{u}}^{(\gamma)} \xi_{\tilde{q}}^{+} \right)_{j+\frac{1}{2}},
\end{aligned}
$$

where we have used the fact

$$
\begin{aligned}
\widehat{f'(\tilde{u})\xi_{\tilde{u}}} &= f'(\tilde{u})\{\!\{\xi_{\tilde{u}}\}\!\} - (\theta f'(\tilde{u}) + \lambda\tilde{\alpha})[\![\xi_{\tilde{u}}]\!] \\
&= f'(\tilde{u})\xi_{\tilde{u}}^{(\gamma)} + \left(\left(\gamma - \theta - \frac{1}{2} \right) f'(\tilde{u}) - \lambda\tilde{\alpha} \right) [\![\xi_{\tilde{u}}]\!].
\end{aligned}
$$

Taking $z_h = \xi_{\tilde{q}}$ in the cell error equation corresponding to (3.10c), we get

$$
\epsilon_0 \mathcal{H}_j^{\gamma}(\xi_{\tilde{u}}, \xi_{\tilde{q}}) = -\int_{I_j} \xi_{\tilde{q}}\xi_{\tilde{q}} \mathrm{d}x - \int_{I_j} \eta_{\tilde{q}}\xi_{\tilde{q}} \mathrm{d}x.
$$

An application of the inverse inequalities leads to the bound for $\mathcal{H}^{\wedge}(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}})$, it reads,

$$
\begin{aligned}
\left| \mathcal{H}^{\wedge}\left(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}} \right) \right| &\le C_\star \|\xi_{\tilde{u}}\| \, \|\xi_{\tilde{q}}\| + \frac{C}{\epsilon_0} \left(\|\xi_{\tilde{q}}\|^2 + h^{k+1}\|\xi_{\tilde{q}}\| \right) \\
&\quad + C \left| \sum_{j=1}^{N} \left([\![\xi_{\tilde{u}}]\!] [\![\xi_{\tilde{q}}]\!] \right)_{j+\frac{1}{2}} \right|.
\end{aligned} \tag{3.14a}
$$

Analogously,

$$
\begin{aligned}
\left| \mathcal{H}^{\wedge}\left(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{p}} \right) \right| &\le C_\star \|\xi_{\tilde{u}}\| \, \|\xi_{\tilde{p}}\| + \frac{C}{\epsilon_0} \left(\|\xi_{\tilde{q}}\| \, \|\xi_{\tilde{p}}\| + h^{k+1}\|\xi_{\tilde{p}}\| \right) \\
&\quad + C \left| \sum_{j=1}^{N} \left([\![\xi_{\tilde{u}}]\!] [\![\xi_{\tilde{p}}]\!] \right)_{j+\frac{1}{2}} \right|.
\end{aligned} \tag{3.14b}
$$

For the estimates of jump terms, taking $(v_h, w_h, z_h) = (\xi_{\tilde{p}}, \xi_{\tilde{q}}, \xi_{\tilde{u}})$ in (3.10), by Lemma 2.1, the properties of $\mathbb{P}_h^{\gamma}, \mathbb{P}_h^{\widetilde{\gamma}}, \mathsf{P}_d^{\widetilde{\gamma}}$ and the estimate of $|\mathcal{H}^{\wedge}(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{p}})|$ in (3.14b), we get

$$
\sum_{j=1}^{N} [\![\xi_{\tilde{u}}]\!]_{j+\frac{1}{2}}^2 \le \frac{C}{\epsilon_0} \left(\|\xi_{\tilde{u}}\| \, \|\xi_{\tilde{q}}\| + h^{k+1}\|\xi_{\tilde{u}}\| \right), \tag{3.15a}
$$

$$
\sum_{j=1}^{N} [\![\xi_{\tilde{q}}]\!]_{j+\frac{1}{2}}^2 \le \frac{C}{\epsilon_0} \left(\|\xi_{\tilde{q}}\| \, \|\xi_{\tilde{p}}\| + h^{k+1}\|\xi_{\tilde{q}}\| \right), \tag{3.15b}
$$

$$
\sum_{j=1}^{N} [\![\xi_{\tilde{p}}]\!]_{j+\frac{1}{2}}^2 \le \frac{C}{\epsilon_0} \left(\|\xi_{\tilde{u}}\| \, \|\xi_{\tilde{p}}\| + \|\xi_{\tilde{q}}\| \, \|\xi_{\tilde{p}}\| + \|\xi_{\tilde{u}}\| \, \|\xi_{\tilde{q}}\| + h^{k+1}(\|\xi_{\tilde{u}}\| + \|\xi_{\tilde{p}}\|) \right), \tag{3.15c}
$$

where we have used Young's inequality for boundary terms of (3.14b) and also (3.15a) in the derivation of (3.15c). We are now ready to estimate $\|\xi_{\tilde{q}}\|$. Taking $(w_h, z_h) = (\xi_{\tilde{u}}, \xi_{\tilde{q}})$ in (3.10b)-(3.10c) and adding them up, we arrive at

$$\|\xi_{\tilde{q}}\|^2 = -\int_I \xi_{\tilde{p}}\xi_{\tilde{u}}\mathrm{d}x - \int_I \eta_{\tilde{p}}\xi_{\tilde{u}}\mathrm{d}x - \int_I \eta_{\tilde{q}}\xi_{\tilde{q}}\mathrm{d}x - \epsilon_0\big(\mathcal{H}^{\widetilde{\mu}}(\xi_{\tilde{q}}, \xi_{\tilde{u}}) + \mathcal{H}^{\gamma}(\xi_{\tilde{u}}, \xi_{\tilde{q}})\big).$$

It follows from Lemma 2.1, the approximation properties of projections, the Young's inequality and the estimate of jump terms in (3.15a)-(3.15b) that

$$\begin{aligned}
\|\xi_{\tilde{q}}\|^2 &\le \|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{p}}\| + Ch^{k+1}\big(\|\xi_{\tilde{q}}\| + \|\xi_{\tilde{u}}\|\big) + \epsilon_0\sum_{j=1}^{N}\left(\frac{(\gamma-\mu)^2}{4\epsilon_2}[\![\xi_{\tilde{u}}]\!]^2_{j+\frac12} + \epsilon_2[\![\xi_{\tilde{q}}]\!]^2_{j+\frac12}\right)\\
&\le \|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{p}}\| + \frac{(\gamma-\mu)^2}{4\epsilon_2}C\|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{q}}\| + C\epsilon_2\|\xi_{\tilde{q}}\|\,\|\xi_{\tilde{p}}\| + \frac{(\gamma-\mu)^2}{4\epsilon_2}Ch^{k+1}\|\xi_{\tilde{u}}\|\\
&\quad + C\epsilon_2h^{k+1}\|\xi_{\tilde{q}}\| + Ch^{k+1}\big(\|\xi_{\tilde{q}}\| + \|\xi_{\tilde{u}}\|\big)\\
&\le \frac{\epsilon_2}{8}\|\xi_{\tilde{p}}\|^2 + \frac{2}{\epsilon_2}\|\xi_{\tilde{u}}\|^2 + \frac18\|\xi_{\tilde{q}}\|^2 + \frac{C^2(\gamma-\mu)^4}{8\epsilon_2^2}\|\xi_{\tilde{u}}\|^2 + \frac{\epsilon_2}{8}\|\xi_{\tilde{p}}\|^2 + 2C^2\epsilon_2\|\xi_{\tilde{q}}\|^2\\
&\quad + \frac18\|\xi_{\tilde{q}}\|^2 + \|\xi_{\tilde{u}}\|^2 + Ch^{2k+2}\\
&\le \left(2C^2\epsilon_2 + \frac14\right)\|\xi_{\tilde{q}}\|^2 + \left(1 + \frac{2}{\epsilon_2} + \frac{C^2(\gamma-\mu)^4}{8\epsilon_2^2}\right)\|\xi_{\tilde{u}}\|^2 + \frac{\epsilon_2}{4}\|\xi_{\tilde{p}}\|^2 + Ch^{2k+2},
\end{aligned}$$

where ϵ_2 can be sufficiently small. Taking $\epsilon_2 = 1/(4C^2)$, it is easy to show that

$$\|\xi_{\tilde{q}}\|^2 \le C_1\|\xi_{\tilde{u}}\|^2 + \epsilon_2\|\xi_{\tilde{p}}\|^2 + Ch^{2k+2}, \tag{3.16}$$

where C_1 depends on C, γ, μ, but is independent of h.

Step 4: Estimate to $\|\xi_{\tilde{p}}\|$. Taking $(v_h, w_h) = (-\xi_{\tilde{q}}, \xi_{\tilde{p}})$ in (3.10a)-(3.10b) and summing them up, we have

$$\begin{aligned}
\|\xi_{\tilde{p}}\|^2 &= \int_I \xi_{\tilde{u}}\xi_{\tilde{q}}\mathrm{d}x + \int_I \eta_{\tilde{u}}\xi_{\tilde{q}}\mathrm{d}x - \int_I \eta_{\tilde{p}}\xi_{\tilde{p}}\mathrm{d}x + \tilde{\epsilon}_0\left(-\mathcal{H}^\wedge\big(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}}\big) + \mathcal{H}^\wedge\big(\tilde{R}e^2_{\tilde{u}}, \xi_{\tilde{q}}\big)\right)\\
&\quad - \tilde{\epsilon}_0\mathcal{H}^\wedge\big(f'(\tilde{u})\eta_{\tilde{u}}, \xi_{\tilde{q}}\big) - \epsilon_0\big(\mathcal{H}^{\widetilde{\mu}}(\xi_{\tilde{q}}, \xi_{\tilde{p}}) + \mathcal{H}^{\widetilde{\gamma}}(\xi_{\tilde{p}}, \xi_{\tilde{q}}) + \mathcal{H}^{\widetilde{\gamma}}(\eta_{\tilde{p}}, \xi_{\tilde{q}})\big).
\end{aligned}$$

By Lemma 2.1, the approximation properties of projections, the assumption $\|u_0 - u_h(0)\|_\infty \le Ch$ together with the estimates of $\mathcal{H}^\wedge(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}})$ and jump terms in (3.14a), (3.15), we obtain, after using Young's inequality

$$\begin{aligned}
\|\xi_{\tilde{p}}\|^2 &\le 2\|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{q}}\| + Ch^{k+1}\big(\|\xi_{\tilde{q}}\| + \|\xi_{\tilde{p}}\|\big) + \epsilon_0\sum_{j=1}^{N}\left(\frac{C^2}{4}[\![\xi_{\tilde{u}}]\!]^2_{j+\frac12} + [\![\xi_{\tilde{q}}]\!]^2_{j+\frac12}\right)\\
&\quad + C\|\xi_{\tilde{q}}\|^2 + \epsilon_0\sum_{j=1}^{N}\left(\frac{(\gamma-\widetilde{\mu})^2}{4}[\![\xi_{\tilde{q}}]\!]^2_{j+\frac12} + [\![\xi_{\tilde{p}}]\!]^2_{j+\frac12}\right)\\
&\le 2\|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{q}}\| + C\big(\|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{q}}\| + \|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{p}}\| + \|\xi_{\tilde{q}}\|\,\|\xi_{\tilde{p}}\|\big) + \frac{(\gamma-\widetilde{\mu})^2}{4}C\|\xi_{\tilde{q}}\|\,\|\xi_{\tilde{p}}\|\\
&\quad + \frac{C^3}{4}\|\xi_{\tilde{u}}\|\,\|\xi_{\tilde{q}}\| + C\|\xi_{\tilde{q}}\|^2 + \frac{(\gamma-\widetilde{\mu})^2}{4}Ch^{k+1}\|\xi_{\tilde{q}}\| + \frac{C^3}{4}h^{k+1}\|\xi_{\tilde{u}}\|\\
&\quad + Ch^{k+1}\big(\|\xi_{\tilde{u}}\| + \|\xi_{\tilde{q}}\| + \|\xi_{\tilde{p}}\|\big)\\
&\le \frac12\|\xi_{\tilde{p}}\|^2 + \frac{C_2}{2}\big(\|\xi_{\tilde{u}}\|^2 + \|\xi_{\tilde{q}}\|^2\big) + \frac{C}{2}h^{2k+2}.
\end{aligned}$$

Consequently,

$$\|\xi_{\tilde p}\|^2 \le C_2\big(\|\xi_{\tilde u}\|^2 + \|\xi_{\tilde q}\|^2\big) + Ch^{2k+2}, \tag{3.17}$$

where C_2 depends on C, γ, μ, but is independent of h. Substituting (3.17) into (3.16) and letting $\epsilon_2 = \min(1/(4C^2), 1/(2C_2))$, it is easy to get

$$\|\xi_{\tilde q}\|^2 \le (2C_1+1)\|\xi_{\tilde u}\|^2 + Ch^{2k+2}, \tag{3.18a}$$

$$\|\xi_{\tilde p}\|^2 \le 2C_2(C_1+1)\|\xi_{\tilde u}\|^2 + Ch^{2k+2}. \tag{3.18b}$$

Step 5: Final estimates. Inserting (3.18) into (3.13) and taking

$$\epsilon_1 = \frac{1}{4\big(2C_1+1+2C_2(C_1+1)\big)},$$

we obtain

$$\|\xi_{\tilde u}\| \le Ch^{k+1}, \quad \|\xi_{\tilde q}\| + \|\xi_{\tilde p}\| \le Ch^{k+1},$$

where C is a constant independent of h. Further, by the relationship (3.8) and the property of the projection $\mathsf{P}_d^{\tilde\gamma}$ in Lemma 3.1, we obtain

$$\|\xi_u(0)\| + \|\xi_q(0)\| + \|\xi_p(0)\| \le Ch^{k+1}.$$

It remains to consider $\|\xi_{u_t}(0)\|$. The error equation for (2.7a) is

$$\begin{aligned}\int_I \xi_{u_t} v_h \mathrm{d}x = &-\int_I \eta_{u_t} v_h \mathrm{d}x + \mathcal{H}^{\wedge}\big(f'(u)\xi_u, v_h\big) + \mathcal{H}^{\wedge}\big(f'(u)\eta_u, v_h\big)\\ &- \mathcal{H}^{\wedge}\big(Re_u^2, v_h\big) + \mathcal{H}^{\tilde\gamma}(\xi_p, v_h) + \mathcal{H}^{\tilde\gamma}(\eta_p, v_h).\end{aligned}$$

Due to the continuity of numerical solutions with respect to time, the above equation still holds when $t=0$. This, by virtue of (3.8) and $\tilde\epsilon_0 = \epsilon_0^3$ in combination with the error equation (3.10a), leads to

$$\int_I \xi_{u_t}(0) v_h \mathrm{d}x = -\int_I \eta_{u_t}(0) v_h \mathrm{d}x + \frac{1}{\tilde\epsilon_0}\int_I e_u v_h \mathrm{d}x. \tag{3.19}$$

Taking $v_h{=}\xi_{u_t}(0)$, it is easy to get $\|\xi_{u_t}(0)\|{\le}Ch^{k+1}$. This finishes the proof of Lemma 3.2. □

4. Optimal Error Estimates

For the LDG scheme (2.7), using the same argument as that in (3.10) for initial error estimates, we arrive at the following error equations:

$$\begin{aligned}\int_I \xi_{u_t} v_h \mathrm{d}x = &-\int_I \eta_{u_t} v_h \mathrm{d}x + \mathcal{H}^{\wedge}\big(f'(u)\xi_u, v_h\big) + \mathcal{H}^{\wedge}\big(f'(u)\eta_u, v_h\big)\\ &- \mathcal{H}^{\wedge}\big(Re_u^2, v_h\big) + \mathcal{H}^{\tilde\gamma}(\xi_p, v_h) + \mathcal{H}^{\tilde\gamma}(\eta_p, v_h),\end{aligned} \tag{4.1a}$$

$$\int_I \xi_p w_h \mathrm{d}x = -\int_I \eta_p w_h \mathrm{d}x - \mathcal{H}^{\tilde\mu}(\xi_q, w_h), \tag{4.1b}$$

$$\int_I \xi_q z_h \mathrm{d}x = -\int_I \eta_q z_h \mathrm{d}x - \mathcal{H}^{\gamma}(\xi_u, z_h), \tag{4.1c}$$

which hold for all $(v_h, w_h, z_h) \in [V_h^k]^3$, where we have used the definitions of projections $\mathbb{P}_h^{\gamma} u, \mathbb{P}_h^{\widetilde{\mu}} q$ in (3.1). To deal with the nonlinearity of the flux $f(u)$ at $t > 0$, we would like to adopt the a priori assumption [26] that for $k \geq 1$,

$$\|\xi_u(t)\| \leq h^{\frac{3}{2}}, \quad \|\xi_{ut}(t)\| \leq h^{\frac{3}{2}}, \tag{4.2a}$$

where $\xi_u = \mathbb{P}_h^{\gamma} u - u_h$ and $\mathbb{P}_h^{\gamma} u$ is a GGR projection of u defined in (3.1) and $t > 0$. By inverse property (2.5), we have

$$\|\xi_u(t)\|_\infty \leq Ch, \quad \|\xi_{ut}(t)\|_\infty \leq Ch. \tag{4.2b}$$

To derive the optimal error estimates, let us first show an intermediate result for $\|\xi_q\|, \|\xi_p\|$.

Lemma 4.1. *The LDG solution to the scheme* (2.7) *with generalized numerical fluxes* (2.8) *satisfies*

$$\|\xi_q\|^2 + \|\xi_p\|^2 \leq C\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2 + h^{2k+2}\big), \tag{4.3}$$

where C is independent of h.

Proof. We start by presenting the estimates to jump terms and $\mathcal{H}^\wedge(f'(u)\xi_u, \xi_q)$ to be used later. Letting $(v_h, w_h, z_h) = (\xi_p, \xi_q, \xi_u)$ in (4.1) and using the same argument as that in deriving (3.15) for initial error estimate, we obtain

$$\sum_{j=1}^{N} [\![\xi_u]\!]_{j+\frac{1}{2}}^2 \leq C\big(\|\xi_u\|\,\|\xi_q\| + h^{k+1}\|\xi_u\|\big), \tag{4.4a}$$

$$\sum_{j=1}^{N} [\![\xi_q]\!]_{j+\frac{1}{2}}^2 \leq C\big(\|\xi_q\|\,\|\xi_p\| + h^{k+1}\|\xi_q\|\big), \tag{4.4b}$$

$$\begin{aligned} \sum_{j=1}^{N} [\![\xi_p]\!]_{j+\frac{1}{2}}^2 \leq{}& C\big(\|\xi_{ut}\|\,\|\xi_p\| + \|\xi_u\|\,\|\xi_p\| + \|\xi_u\|\,\|\xi_q\| + \|\xi_q\|\,\|\xi_p\|\big) \\ &+ Ch^{k+1}\big(\|\xi_u\| + \|\xi_p\|\big). \end{aligned} \tag{4.4c}$$

Similar to the estimate of $\mathcal{H}^\wedge(f'(\tilde{u})\xi_{\tilde{u}}, \xi_{\tilde{q}})$ in (3.14a), we have

$$\big|\mathcal{H}^\wedge\big(f'(u)\xi_u, \xi_q\big)\big| \leq C\big(\|\xi_u\|^2 + \|\xi_q\|^2 + h^{2k+2}\big) + C\left|\sum_{j=1}^{N} \big([\![\xi_u]\!][\![\xi_q]\!]\big)_{j+\frac{1}{2}}\right|. \tag{4.5}$$

We are now ready to estimate $\|\xi_q\|$ and $\|\xi_p\|$. Letting $(w_h, z_h) = (\xi_u, \xi_q)$ in (4.1b)-(4.1c) and adding them up, we arrive at

$$\|\xi_q\|^2 = -\int_I \xi_p \xi_u \mathrm{d}x - \int_I \eta_p \xi_u \mathrm{d}x - \int_I \eta_q \xi_q \mathrm{d}x - \mathcal{H}^{\widetilde{\mu}}(\xi_q, \xi_u) - \mathcal{H}^{\gamma}(\xi_u, \xi_q).$$

By virtue of (4.4a)-(4.4b) and using an analogous argument as that of $\|\xi_{\tilde{q}}\|^2$ in the initial error estimate, we get

$$\begin{aligned} \|\xi_q\|^2 &\leq \|\xi_u\|\,\|\xi_p\| + Ch^{k+1}\big(\|\xi_q\| + \|\xi_u\|\big) + \sum_{j=1}^{N} \left(\frac{(\gamma-\mu)^2}{4\epsilon}[\![\xi_u]\!]_{j+\frac{1}{2}}^2 + \epsilon[\![\xi_q]\!]_{j+\frac{1}{2}}^2\right) \\ &\leq \|\xi_u\|\,\|\xi_p\| + \frac{(\gamma-\mu)^2}{4\epsilon}C\|\xi_u\|\,\|\xi_q\| + C\epsilon\|\xi_q\|\,\|\xi_p\| + \frac{(\gamma-\mu)^2}{4\epsilon}Ch^{k+1}\|\xi_u\| \end{aligned}$$

$$
\begin{aligned}
&+ C\epsilon h^{k+1}\|\xi_q\| + Ch^{k+1}\big(\|\xi_q\| + \|\xi_u\|\big) \\
\le\; & \frac{\epsilon}{8}\|\xi_p\|^2 + \frac{2}{\epsilon}\|\xi_u\|^2 + \frac{1}{8}\|\xi_q\|^2 + \frac{C^2(\gamma-\mu)^4}{8\epsilon^2}\|\xi_u\|^2 + \frac{\epsilon}{8}\|\xi_p\|^2 + 2C^2\epsilon\|\xi_q\|^2 \\
&+ \frac{1}{8}\|\xi_q\|^2 + \|\xi_u\|^2 + Ch^{2k+2} \\
\le\; & \left(2C^2\epsilon + \frac{1}{4}\right)\|\xi_q\|^2 + \left(1 + \frac{2}{\epsilon} + \frac{C^2(\gamma-\mu)^4}{8\epsilon^2}\right)\|\xi_u\|^2 + \frac{\epsilon}{4}\|\xi_p\|^2 + Ch^{2k+2},
\end{aligned}
$$

where ϵ obtained by the Young's inequality satisfies $2C^2\epsilon \le 1/2$. Thus,

$$
\|\xi_q\|^2 \le C_1\|\xi_u\|^2 + \epsilon\|\xi_p\|^2 + Ch^{2k+2}. \tag{4.6}
$$

Taking $(v_h, w_h) = (-\xi_q, \xi_p)$ in error equations (4.1a)-(4.1b) and adding them up, we have

$$
\begin{aligned}
\|\xi_p\|^2 =& \int_I \xi_{ut}\xi_q \mathrm{d}x + \int_I \eta_{ut}\xi_q \mathrm{d}x - \int_I \eta_p\xi_p \mathrm{d}x - \mathcal{H}^{\wedge}\big(f'(u)\xi_u, \xi_q\big) + \mathcal{H}^{\wedge}\big(Re_u^2, \xi_q\big) \\
&- \mathcal{H}^{\wedge}\big(f'(u)\eta_u, \xi_q\big) - \mathcal{H}^{\widetilde{\gamma}}(\xi_p, \xi_q) - \mathcal{H}^{\widetilde{\mu}}(\xi_q, \xi_p) - \mathcal{H}^{\widetilde{\gamma}}(\eta_p, \xi_q).
\end{aligned}
$$

Collecting (4.4), (4.5) and following the analogous argument as that of $\|\xi_{\tilde{p}}\|^2$ in the initial error estimate, one has

$$
\|\xi_p\|^2 \le C_2\big(\|\xi_u\|^2 + \|\xi_q\|^2 + \|\xi_{ut}\|^2\big) + Ch^{2k+2}. \tag{4.7}
$$

Substituting (4.7) into (4.6) and letting $\epsilon = \min(1/(4C^2), 1/(2C_2))$, we deduce that

$$
\|\xi_q\|^2 + \|\xi_p\|^2 \le C\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2 + h^{2k+2}\big).
$$

This completes the proof. □

Theorem 4.1. *Assume that the exact solution u of* (1.1) *is sufficiently smooth, i.e.* $\|u\|_{k+2}$, $\|u_t\|_{k+2}$, $\|u_{tt}\|_{k+2}$ *are bounded uniformly for any time t, and $f \in C^2$. Let u_h, q_h, p_h be the LDG solution of* (2.7) *with generalized numerical fluxes* (2.8)*. For a quasi-uniform mesh and $k \ge 1$, we have, for any $t > 0$, the following optimal error estimates:*

$$
\|e_u(t)\| + \|e_{ut}(t)\| + \|e_q(t)\| + \|e_p(t)\| \le Ch^{k+1}, \tag{4.8}
$$

where C is independent of h.

Proof. Taking $(v_h, z_h) = (\xi_u, -\xi_p)$ in (4.1a), (4.1c) and adding them up, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5,
$$

where

$$
\begin{aligned}
\Pi_1 &= \int_I \xi_q\xi_p \mathrm{d}x - \int_I \eta_{ut}\xi_u \mathrm{d}x + \int_I \eta_q\xi_p \mathrm{d}x, \\
\Pi_2 &= \mathcal{H}^{\widetilde{\gamma}}(\xi_p, \xi_u) + \mathcal{H}^{\gamma}(\xi_u, \xi_p), \\
\Pi_3 &= \mathcal{H}^{\wedge}\big(f'(u)\xi_u, \xi_u\big), \\
\Pi_4 &= \mathcal{H}^{\wedge}\big(f'(u)\eta_u, \xi_u\big) + \mathcal{H}^{\widetilde{\gamma}}(\eta_p, \xi_u), \\
\Pi_5 &= -\mathcal{H}^{\wedge}\big(Re_u^2, \xi_u\big).
\end{aligned}
$$

Following the process (3.12a)-(3.12e) in Lemma 3.2, it is easy to show

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 \le C\big(\|\xi_u\|^2+\|\xi_q\|^2+\|\xi_p\|^2+h^{2k+2}\big). \tag{4.9a}$$

Taking the time derivatives of (4.1), choosing $(v_h, w_h, z_h) = (\xi_{u_t}, \xi_{q_t}, -\xi_{p_t})$ and adding them up, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_{u_t}\|^2 = W_1+W_2+W_3+W_4+W_5,$$

where

$$\begin{aligned}
W_1 &= -\int_I (\eta_u)_{tt}\xi_{u_t}\mathrm{d}x - \int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x,\\
W_2 &= \mathcal{H}^{\widetilde{\gamma}}(\xi_{p_t},\xi_{u_t}) + \mathcal{H}^{\gamma}(\xi_{u_t},\xi_{p_t}) - \mathcal{H}^{\widetilde{\mu}}(\xi_{q_t},\xi_{q_t}),\\
W_3 &= \mathcal{H}^{\wedge}\big((f'(u)\xi_u)_t,\xi_{u_t}\big),\\
W_4 &= \mathcal{H}^{\wedge}\big((f'(u)\eta_u)_t,\xi_{u_t}\big) + \mathcal{H}^{\widetilde{\gamma}}(\eta_{p_t},\xi_{u_t}),\\
W_5 &= -\mathcal{H}^{\wedge}\big((Re_u^2)_t,\xi_{u_t}\big).
\end{aligned}$$

Using the same argument as that in the estimate of (3.11) except for W_3, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_{u_t}\|^2 \le W_3 - \int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x + C\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2+h^{2k+2}\big). \tag{4.9b}$$

A combination of (4.9a) and (4.9b) together with Lemma 4.1 leads to

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2\big) &\le W_3 - \int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x\\
&\quad + C\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2+\|\xi_q\|^2+\|\xi_p\|^2+h^{2k+2}\big)\\
&\le W_3 - \int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x\\
&\quad + C\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2+h^{2k+2}\big).
\end{aligned} \tag{4.10}$$

Let us now consider the estimate of W_3. Since

$$W_3 = \mathcal{H}^{\wedge}\big(f'(u)\xi_{u_t},\xi_{u_t}\big) + \mathcal{H}^{\wedge}\big(f'(u)_t\xi_u,\xi_{u_t}\big) \triangleq Z_1+Z_2,$$

we use the same argument as that for (3.12c) to obtain

$$Z_1 \le C\|\xi_{u_t}\|^2 - \sum_{j=1}^{N}\big(\theta f'(u)+\lambda\alpha\big)_{j+\frac{1}{2}}[\![\xi_{u_t}]\!]^2_{j+\frac{1}{2}},$$

where $(\theta f'(u)+\lambda\alpha)_{j+1/2} \ge C_0$ by the definition of α and $\lambda \ge |\theta|$, and C_0 is a positive constant independent of h. As for Z_2, we follow the estimate of (3.14a) to get

$$Z_2 \le \sum_{j=1}^{N}\left(\left(\gamma-\theta-\frac{1}{2}\right)f'(u)_t - \lambda\alpha_t\right)_{j+\frac{1}{2}}\big([\![\xi_u]\!][\![\xi_{u_t}]\!]\big)_{j+\frac{1}{2}} + C\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2+h^{2k+2}\big).$$

By Young's inequality, the estimate of $[\![\xi_u]\!]$ in (4.4a) and Lemma 4.1, we arrive at the estimate of Z_2

$$Z_2 \le C\big(\|\xi_u\|^2+\|\xi_{u_t}\|^2+h^{2k+2}\big) + \frac{1}{4}\sum_{j=1}^{N}\big(\theta f'(u)+\lambda\alpha\big)_{j+\frac{1}{2}}[\![\xi_{u_t}]\!]^2_{j+\frac{1}{2}}.$$

Consequently,

$$\begin{aligned} W_3 &\le C\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2 + h^{2k+2}\big) - \frac{3}{4}\big(\theta f'(u) + \lambda\alpha\big)_{j+\frac{1}{2}}[\![\xi_{ut}]\!]^2_{j+\frac{1}{2}} \\ &\le C\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2 + h^{2k+2}\big). \end{aligned} \tag{4.11}$$

Inserting the above estimate into (4.10), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2\big) \le C\big(\|\xi_u\|^2 + \|\xi_{ut}\|^2 + h^{2k+2}\big) - \int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x.$$

Integrating the above inequality from 0 to T and using integration by parts in time together with the initial error estimates in Lemma 3.2, we get

$$\begin{aligned} \|\xi_u(T)\|^2 + \|\xi_{ut}(T)\|^2 &\le C\int_0^T \big(\|\xi_u\|^2 + \|\xi_{ut}\|^2\big)\mathrm{d}t + \|\xi_u(0)\|^2 + \|\xi_{ut}(0)\|^2 \\ &\quad + 2\int_0^T \Big(-\int_I \eta_{p_t}\xi_{q_t}\mathrm{d}x + \int_I \eta_{q_t}\xi_{p_t}\mathrm{d}x\Big)\mathrm{d}t + Ch^{2k+2} \\ &\le C\int_0^T \big(\|\xi_u\|^2 + \|\xi_{ut}\|^2\big)\mathrm{d}t + 2\int_0^T\int_I \big(\eta_{p_{tt}}\xi_q - \eta_{q_{tt}}\xi_p\big)\mathrm{d}x\mathrm{d}t \\ &\quad + 2\int_I \big(-\eta_{p_t}\xi_q + \eta_{q_t}\xi_p\big)\big|_0^T \mathrm{d}x + Ch^{2k+2}, \\ &\le C\int_0^T \big(\|\xi_u\|^2 + \|\xi_{ut}\|^2\big)\mathrm{d}t + \frac{1}{2}\big(\|\xi_u(T)\|^2 + \|\xi_{ut}(T)\|^2\big) + Ch^{2k+2}, \end{aligned}$$

where we have also used the optimal approximation property of projections, Young's inequality and Lemma 4.1. An application of Gronwall's inequality and the approximation property of the projection leads to the desired optimal error estimate

$$\|e_u(t)\| + \|e_{ut}(t)\| + \|e_q(t)\| + \|e_p(t)\| \le Ch^{k+1}.$$

Finally, it follows from the above inequality that

$$\|\xi_u(t)\| + \|\xi_{ut}(t)\| \le Ch^{k+1} \le Ch^2 \le h^{\frac{3}{2}},$$

and thus the a priori assumption (4.2a) is reasonable. This finishes the proof. □

5. Numerical Experiments

In this section, numerical experiments including accuracy tests, uniform and nonuniform meshes (10% random perturbation of the uniform mesh), lower growth of the error, performance for mixed boundary conditions and the capacity in resolving waves are presented. The explicit third order total variation diminishing Runge-Kutta time discretization is used. We would like to remark that the parameter $\tilde{\epsilon}_0$ in (2.9) is designed for the technical purpose only in deriving optimal initial error estimate for the nonlinear case. In actual numerical computations, we can simply use the standard L^2 projection as the numerical initial condition and still observe expected optimal convergence orders.

Example 5.1. Consider

$$\begin{aligned} &u_t + (3u^2)_x + u_{xxx} = g(x,t), && (x,t) \in [0,\pi] \times (0,T], \\ &u(x,0) = \sin(2x), && x \in [0,\pi] \end{aligned}$$

with periodic boundary conditions. The source term $g(x,t)$ is chosen such that the exact solution is

$$u(x,t) = \sin(2x+t).$$

Table 5.1 lists the L^2 errors and numerical orders for Example 5.1 with different θ, γ, μ and λ at $T = 1$, in which P^k polynomials with $0 \le k \le 3$ and uniform meshes are used. The results with nonuniform meshes are shown in Table 5.2, and we can always observe optimal $(k+1)$-th order, confirming the sharpness of the result in Theorem 4.1.

Example 5.2. To show the capacity in resolving waves with mixed boundary conditions, consider the double solitary-wave problem

$$u_t + (3u^2)_x + u_{xxx} = 0, \quad (x,t) \in [-20,1] \times (0,T]$$

with the initial condition

$$u(x,0) = \frac{45\text{csch}^2\big(1.5(x+14.5)\big) + 20\text{sech}^2(x+12)}{2\big(3\coth\big(1.5(x+14.5)\big) - 2\tanh(x+12)\big)^2}$$

Table 5.1: The errors $\|u - u_h\|$ and orders for Example 5.1 using P^k polynomials with different θ, γ, μ and λ on a uniform mesh of N cells. $T = 1$.

	N	$\theta=0.4$, $\gamma=0.7$, $\mu=0.9$, $\lambda=0.5$		$\theta=0.1$, $\gamma=0.6$, $\mu=0.8$, $\lambda=-0.1$		$\theta=0.0$, $\gamma=1.0$, $\mu=1.0$, $\lambda=0.5$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	5.68E-01	–	1.63E-01	–	5.65E-01	–
	40	2.98E-01	0.93	6.61E-02	1.30	2.99E-01	0.92
	80	1.31E-01	1.18	3.16E-02	1.06	1.32E-01	1.18
	160	5.47E-02	1.26	1.57E-02	1.01	5.51E-02	1.26
P^1	20	1.48E-02	–	2.38E-02	–	7.57E-03	–
	40	3.48E-03	1.94	7.02E-03	1.76	1.88E-03	2.00
	80	9.71E-04	1.99	1.85E-03	1.93	4.71E-04	2.00
	160	2.43E-04	2.00	4.68E-04	1.98	1.18E-04	2.00
P^2	20	1.37E-04	–	1.27E-04	–	1.89E-04	–
	40	1.70E-05	3.01	1.58E-05	3.01	2.37E-05	2.99
	80	2.12E-06	3.00	1.97E-06	3.00	2.96E-06	3.00
	160	2.65E-07	3.00	2.46E-07	3.00	3.70E-07	3.00
P^3	20	6.78E-06	–	1.10E-05	–	3.65E-06	–
	40	4.47E-07	3.92	8.13E-07	3.76	2.29E-07	3.99
	60	8.93E-08	3.97	1.67E-07	3.91	4.55E-08	3.99
	80	2.87E-08	3.94	5.37E-08	3.94	1.51E-08	3.83

Table 5.2: The errors $\|u - u_h\|$ and orders for Example 5.1 using P^k polynomials with different θ, γ, μ and λ on a nonuniform mesh of N cells. $T = 1$.

	N	$\theta = 0.4$, $\gamma = 0.7$, $\mu = 0.9$, $\lambda = 0.5$		$\theta = 0.1$, $\gamma = 0.6$, $\mu = 0.8$, $\lambda = -0.1$		$\theta = 0.0$, $\gamma = 1.0$, $\mu = 1.0$, $\lambda = 0.5$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	5.75E-01	–	1.73E-01	–	5.70E-01	–
	40	3.01E-01	0.93	6.90E-02	1.33	3.01E-01	0.92
	80	1.34E-01	1.17	3.39E-02	1.03	1.33E-01	1.18
	160	5.57E-02	1.27	1.69E-02	1.01	5.56E-02	1.25
P^1	20	1.48E-02	–	2.45E-02	–	7.88E-03	–
	40	3.89E-03	1.93	7.07E-03	1.79	1.91E-03	2.05
	80	9.82E-04	1.99	1.87E-03	1.92	4.91E-04	1.96
	160	2.46E-04	2.00	4.73E-04	1.98	1.22E-04	2.01
P^2	20	1.64E-04	–	1.36E-04	–	2.15E-04	–
	40	1.92E-05	3.09	2.23E-05	2.61	2.50E-05	3.11
	80	2.68E-06	2.85	2.62E-06	3.09	3.10E-06	3.01
	160	3.00E-07	3.16	3.38E-07	2.95	3.89E-07	3.00
P^3	20	7.02E-06	–	1.21E-05	–	3.93E-06	–
	40	4.78E-07	3.87	8.41E-07	3.84	2.50E-07	3.98
	60	9.44E-08	4.00	1.73E-07	3.90	5.14E-08	3.90
	80	3.01E-08	3.97	5.65E-08	3.89	1.67E-08	3.92

and mixed boundary conditions

$$\begin{aligned}
u(-20,t) &= \frac{45\mathrm{csch}^2\big(1.5(-5.5-9t)\big)+20\mathrm{sech}^2(-8-4t)}{2\big(3\coth\big(1.5(-5.5-9t)\big)-2\tanh(-8-4t)\big)^2},\\
u(1,t) &= \frac{45\mathrm{csch}^2\big(1.5(15.5-9t)\big)+20\mathrm{sech}^2(13-4t)}{2\big(3\coth\big(1.5(15.5-9t)\big)-2\tanh(13-4t)\big)^2},\\
u_x(1,t) &= \left(\frac{45\mathrm{csch}^2\big(1.5(x+14.5-9t)\big)+20\mathrm{sech}^2(x+12-4t)}{2\big(3\coth\big(1.5(x+14.5-9t)\big)-2\tanh(x+12-4t)\big)^2}\right)_x\Bigg|_{x=1},
\end{aligned}$$

and the exact solution is

$$u(x,t) = \frac{45\mathrm{csch}^2\big(1.5(x+14.5-9t)\big)+20\mathrm{sech}^2(x+12-4t)}{2\big(3\coth\big(1.5(x+14.5-9t)\big)-2\tanh(x+12-4t)\big)^2}.$$

The L^2 errors and numerical orders for Example 5.2 with different θ, γ, μ and λ at $T = 0.5$ are given in Table 5.3. From the table, we observe that optimal $(k+1)$-th order can be always achieved, indicating that the optimal error estimate is also valid for mixed boundary conditions. Moreover, to show the collisions and interactions of soliton waves, we display the graphs of exact solutions and numerical solutions at different time in Fig. 5.1, in which the numerical solutions are computed by using P^2 polynomials with $\theta = -0.4, \gamma = 0.9, \mu = 0.9, \lambda = 0$ and 100 cells. We can see that the two waves collide with each other at about $T = 0.5$ and the shape of waves remains unchanged after collision.

Table 5.3: The errors $\|u - u_h\|$ and orders for Example 5.2 using P^k polynomials with different θ, γ, μ and λ on a uniform mesh of N cells. $T = 0.5$.

	N	$\theta = -0.2$ $\gamma = 0.8$ $\mu = 1.0$ $\lambda = 0.5$		$\theta = -0.4$ $\gamma = 0.9$ $\mu = 0.9$ $\lambda = 0.1$		$\theta = 0.5$ $\gamma = 1.0$ $\mu = 1.0$ $\lambda = 0.0$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^1	100	2.15E-01	–	2.44E-02	–	1.36E-01	–
	120	1.36E-01	2.54	1.62E-02	2.25	8.19E-02	2.79
	140	9.00E-02	2.66	1.16E-02	2.18	5.27E-02	2.85
	160	6.24E-02	2.74	8.71E-03	2.14	3.59E-02	2.89
P^2	100	1.20E-03	–	8.66E-04	–	1.30E-03	–
	120	5.71E-04	4.06	4.95E-04	3.07	6.55E-04	3.77
	140	3.22E-04	3.71	3.09E-04	3.04	3.82E-04	3.50
	160	2.03E-04	3.47	2.06E-04	3.03	2.45E-04	3.32
P^3	60	3.96E-04	–	3.41E-04	–	3.61E-04	–
	80	1.26E-04	3.98	1.12E-04	3.86	1.06E-04	4.27
	100	5.35E-05	3.83	4.69E-05	3.90	4.28E-05	4.05
	120	2.66E-05	3.83	2.29E-05	3.92	2.06E-05	4.00
P^4	20	1.71E-02	–	8.80E-03	–	1.43E-02	–
	40	1.50E-04	6.83	1.24E-04	6.15	1.23E-04	6.85
	60	1.59E-05	5.54	1.65E-05	4.98	1.64E-05	4.97
	80	3.75E-06	5.63	3.61E-06	5.28	3.91E-06	4.99

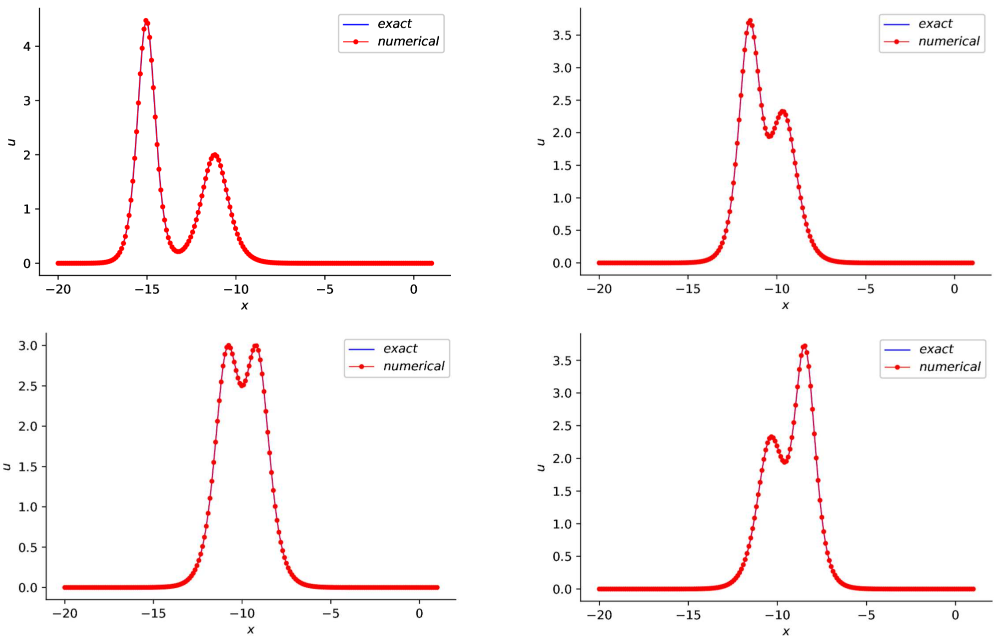

Fig. 5.1. Interactions of the double solitary-wave problem for Example 5.2 with P^2 polynomials and 100 cells, $\theta = -0.4, \gamma = 0.9, \mu = 0.9, \lambda = 0$. Top left: $T = 0$. Top right: $T = 0.4$. Bottom left: $T = 0.5$. Bottom right: $T = 0.6$.

Example 5.3. To illustrate long time behaviors, consider the classical solitary-wave solution of the nonlinear KdV equation

$$u_t + uu_x + \epsilon u_{xxx} = 0, \quad (x,t) \in [0,1] \times (0,T]$$

with the initial condition

$$u(x,0) = A\,\mathrm{sech}^2\big(K(x - x_0)\big),$$

and the exact solution is

$$u(x,t) = A\,\mathrm{sech}^2\big(K(x - x_0 - vt)\big),$$

where $\epsilon = 5 \times 10^{-4}$, $A = 0.9$, $v = A/3$, $K = \sqrt{A/(3\epsilon)}/2$, and $x_0 = 0.5$. Due to exponential decay, it can be treated as periodic boundary conditions.

Fig. 5.2 displays the error curves with classical upwind and alternating fluxes as well as generalized numerical fluxes up to $T = 50$. We use P^2 polynomials with 80 cells. From the figure, we can see that the growth of errors with generalized numerical fluxes ($\theta = -0.3, \gamma = 1.0$, $\mu = 0.7, \lambda = -0.1$ or $\theta = -0.4, \gamma = 0.9, \mu = 0.9, \lambda = 0$) is much lower than that with classical upwind and alternating fluxes ($\theta = 0.5, \gamma = 1.0, \mu = 1.0, \lambda = 0$). Especially, when $\theta = -0.4, \gamma = 0.9, \mu = 0.9, \lambda = 0$, the numerical flux $\hat{f}(u_h)$ is downwind-biased, resulting in an anti-dissipation mechanism to compensate the numerical dissipation of the dispersive term. This example illustrates that the LDG scheme with generalized fluxes is beneficial for long time simulations, when some suitable weights are chosen.

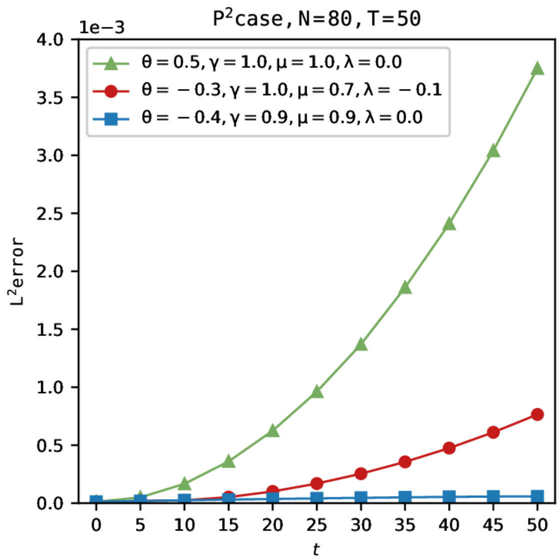

Fig. 5.2. Time history of the L^2 error for Example 5.3, P^2 polynomials with $N = 80$.

Example 5.4. In this example, consider the cnoidal-wave solution of the nonlinear KdV equation

$$u_t + uu_x + \epsilon u_{xxx} = 0, \quad (x,t) \in [0,1] \times (0,T]$$

with periodic boundary conditions and the initial condition

$$u(x,0) = a\,\mathrm{cn}^2\big(4K(x - x_0)\big).$$

The exact solution is

$$u(x,t) = a\,\mathrm{cn}^2\big(4K(x - x_0 - vt)\big),$$

where $\epsilon = 1/24^2, a = 192m\epsilon K^2(m), v = 64\epsilon(2m-1)K^2(m), m = 0.9, x_0 = 0.5$. The function $K = K(m)$ is the complete elliptic integral of the first kind, and the function $\mathrm{cn}(z) = \mathrm{cn}(z : m)$ is the Jacobi elliptic function with modulus m.

Fig. 5.3 shows the pointwise values of LDG solutions with generalized numerical fluxes for a long time T and the exact solution is also provided as a reference. We use P^1 and P^2 polynomials at $T = 25$. We see that the numerical solutions with classical upwind and alternating fluxes $\theta = 0.5, \gamma = 1.0, \mu = 1.0, \lambda = 0$ exhibit visible phase errors, whereas the generalized numerical fluxes with suitable weights can produce a satisfactory wave resolution for long time simulations.

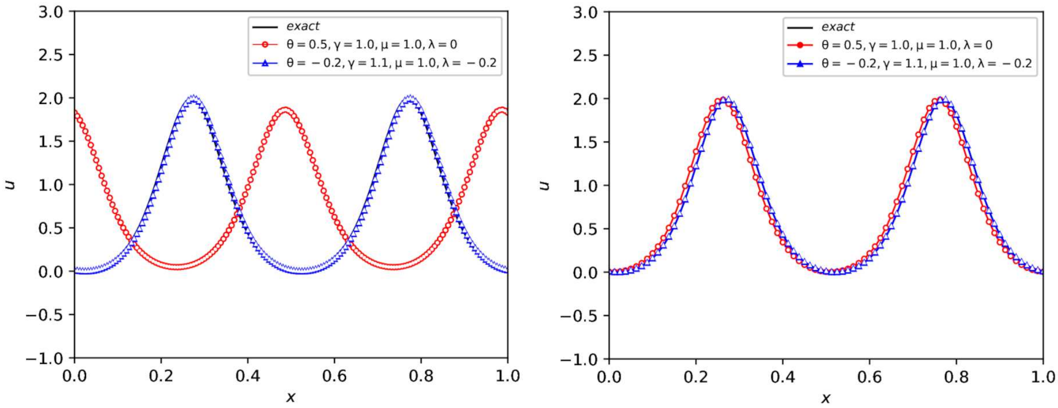

Fig. 5.3. Numerical solutions of the cnoidal-wave for Example 5.4 with different weights. Left: P^1 polynomials with 80 cells at $T = 25$. Right: P^2 polynomials with 40 cells at $T = 25$.

6. Concluding Remarks

We derive optimal error estimates of the LDG method with generalized numerical fluxes for nonlinear KdV type equations. The main difficulty for the treatment of the nonlinear term lies in the construction of modified projections and the relationship between the prime variable and auxiliary variables. The choice of downwind-biased GLF flux for nonlinear convection term will produce an anti-dissipation property to compensate the numerical dissipation of the dispersive term, resulting in a nearly energy conservative scheme with lower error growth and better wave resolution. The validity of the theoretical results is verified by numerical experiments. In future work, we will concentrate on the multi-dimensional case.

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