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A POSTERIORI ERROR ANALYSIS OF THE PML FINITE VOLUME METHOD FOR THE SCATTERING PROBLEM BY A PERIODIC CHIRAL STRUCTURE*

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Abstract

In this paper, we consider the electromagnetic wave scattering problem from a periodic chiral structure. The scattering problem is simplified to a two-dimensional problem, and is discretized by a finite volume method combined with the perfectly matched layer (PML) technique. A residual-type a posteriori error estimate of the PML finite volume method is analyzed and the upper and lower bounds on the error are established in the H^1-norm. The crucial part of the a posteriori error analysis is to derive the error representation formula and use a L^2-orthogonality property of the residual which plays a similar role as the Galerkin orthogonality. An adaptive PML finite volume method is proposed to solve the scattering problem. The PML parameters such as the thickness of the layer and the medium property are determined through sharp a posteriori error estimate. Finally, numerical experiments are presented to illustrate the efficiency of the proposed method.

Mathematics subject classification: 65N08, 65L60, 65N15, 35Q60.
Key words: Finite volume method, Perfectly matched layer, A posteriori error analysis, Chiral media.

1. Introduction

Consider a time-harmonic electromagnetic plane wave incident on a periodic chiral structure. The chiral structure is assumed to be periodic in x_1 direction and invariant in x_2 direction. The medium inside the structure is chiral, and two regions with homogeneous medium are separated by the periodic structure. From the point view of mathematical modeling, our discussion on the scattering problem is simplified to the two-dimensional case. Recently, there has been still a considerable interest in the study of electromagnetic wave propagation by periodic chiral structure. In general, the electromagnetic wave propagation inside the chiral medium are governed by Maxwell equations together with the Drude-Born-Fedorov constitutive equations

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in which the electric and magnetic fields are coupled. The property of the chiral media is completely characterized by the chirality admittance β, the electric permittivity ε and the magnetic permittivity μ. On the other hand, periodic structures have generated great scientific interests in the past several years because of important applications in integrated optics, optical lenses, anti-reflective structures, lasers and so on.

Over the past two decades, scattering problem in chiral structures has gained a great development in the applied mathematical community. For the physical background and the model equations of the scattering problem inside chiral media, many literatures have discussed these issues and we refer to [1, 20, 23, 24, 34] on periodic and non-periodic chiral structures. From the computational aspects, lots of results and references on solving the chiral grating problem may be found in [1, 2, 35, 36]. For other related mathematical analysis and numerical methods of periodic achiral structures, the reader is referred to [3, 6, 15, 17, 18, 28] and references therein.

One of the difficulties for solving the scattering problem is to truncate the unbounded domain into a bounded computational domain with some adequate approximation accuracy. A popular and effective technique in truncating the unbounded domain is the perfectly matched layer (PML) method proposed by Berenger [8]. The key idea of the PML technique is to surround the computational domain by a special designed layer of finite thickness which can make the outgoing waves decay exponentially. At this point, a variety of PML methods have been developed and studied in the literature (cf. [25, 31]). Another difficulty for solving the scattering problem is to deal with the singularities of the solutions, an economical and effective method is the adaptive finite element method based on the a posteriori error estimate (cf. [5, 11, 13, 26, 27]). By using the PML technique in combination, the field of the adaptive finite element method attracted many researchers and has become more and more active in the numerical simulation of the scattering problem, we can refer to [7, 12, 14–16, 21, 22, 32] and references therein for the adaptive PML finite element methods and the related methods. The adaptive finite element methods combined with DtN or PML techniques are very attractive in solving the scattering problems, largely for this reason that DtN or PML method is applied to deal with the difficulty in truncating the unbounded domain and the adaptive finite element method can very efficiently capture the local singularities. However, to our best knowledge, there are very few works on the adaptive DtN or PML finite volume method for solving differential equations. For the literature, there are also some representative results on the posteriori error estimates and the adaptive computations of the finite volume method, the reader is referred to [9, 10, 19, 33] and references therein.

In this paper, we shall study the residual-type a posteriori error estimate of the PML finite volume method (PML-FVM) for solving 1D chiral grating problem. As the PML finite element method (PML-FEM) in [36], our PML finite volume method needs to surround the computational domain by a specially designed artificial layer which absorbs all waves coming from the computational domain. Meanwhile, compared with the DtN finite element method (DtN-FEM), our method can avoid dense blocks of the stiffness matrix generated by the computation of the discrete DtN operator. In this work, the a posteriori error estimate, which includes the finite volume discretization error and the PML error, is established by using similar arguments as the a posteriori error estimate of the PML-FEM. The main difficulty of our error analysis is that our PML-FVM is lack of the global Galerkin orthogonality in contrast to the DtN-FEM and PML-FEM. We overcome this difficulty by using an L^2-orthogonality property of the residual which plays a similar role as the Galerkin orthogonality. The error estimate is used to design the adaptive PML-FVM to choose elements for refinement and to determine the PML parameters and the medium property. Furthermore, the PML error decays exponentially with respect to the distance to the boundary of the fixed domain where the PML layer is placed. Like the adaptive PML-FEM in [15], our adaptive PML-FVM also has the ability to produce coarse mesh size away from the fixed domain and make the total computational costs insensitive to the thickness of the PML absorbing layer. And the lower bound, which shows the efficiency of the a posteriori error estimate, is proved by using the bubble functions. In the last section, we report numerical experiments to demonstrate the feasibility of our adaptive PML-FVM.

The rest of this paper is organized as follows. In Section 2, we introduce the model problem and its variational formulation with the PML boundary condition. In Section 3, we present the finite element discretization and the finite volume discretization for the scattering problem. In Section 4, we analyze the residual-type a posteriori error estimate of the PML-FVM and derive the global upper and local lower bounds of the error which lay down the basis of the adaptive algorithm. In Section 5, we present numerical examples to show the effectiveness of the proposed adaptive algorithm.

2. The Model Problem and the Problem Formulations

In this section, we present a mathematical model for the scattering problem, its DtN formulation and its PML formulation.

2.1. The scattering problem

We consider an adaptive finite volume method for the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$)

$$\begin{aligned} &\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0, \\ &\nabla \times \mathbf{H} + i\omega \mathbf{D} = 0, \end{aligned} \tag{2.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, and $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic displacement vectors in $\mathbb{R}^3$ respectively. In addition, the Drude-Born-Fedorov constitutive equations satisfied by $\mathbf{E}, \mathbf{H}, \mathbf{D}$ and $\mathbf{B}$ can be stated as follows:

$$\begin{aligned} &\mathbf{D} = \varepsilon(x)(\mathbf{E} + \beta(x)\nabla \times \mathbf{E}), \\ &\mathbf{B} = \mu(x)(\mathbf{H} + \beta(x)\nabla \times \mathbf{H}), \end{aligned} \tag{2.2}$$

where $x = (x_1, x_2, x_3)$, and ε, μ and β denote the electric permittivity, the magnetic permeability and the chirality admittance, respectively. After eliminating $\mathbf{D}$ and $\mathbf{B}$, we can reduce (2.1) to the following equations:

$$\begin{aligned} &\nabla \times \mathbf{E} = \big(\gamma(x)\big)^2 \beta(x)\mathbf{E} + i\omega\mu(x)\left(\frac{\gamma(x)}{k(x)}\right)^2 \mathbf{H}, \\ &\nabla \times \mathbf{H} = \big(\gamma(x)\big)^2 \beta(x)\mathbf{H} - i\omega\varepsilon(x)\left(\frac{\gamma(x)}{k(x)}\right)^2 \mathbf{E}, \end{aligned} \tag{2.3}$$

where $k(x)$ and $\gamma(x)$ are defined respectively as

$$k(x) = \omega\sqrt{\varepsilon(x)\mu(x)}, \quad \big(\gamma(x)\big)^2 = \frac{\big(k(x)\big)^2}{1 - \big(k(x)\beta(x)\big)^2}.$$

Through the article, it is assumed that $(k(x)\beta(x))^2 \neq 1$ for $x \in \mathbb{R}^3$. In addition, we assume that the structure is L-periodic with respect to x_1 and invariant with respect to x_2. So it naturally holds that

$$\begin{aligned}\varepsilon(x_1+nL, x_3) &= \varepsilon(x_1, x_3),\\ \mu(x_1+nL, x_3) &= \mu(x_1, x_3),\\ \beta(x_1+nL, x_3) &= \beta(x_1, x_3),\end{aligned}$$

and $\mathbf{E}$ and $\mathbf{H}$ only depend on two variables x_1 and x_3. Since the medium is homogeneous away from a region $\{(x_1, x_3):\ b_2 < x_3 < b_1\}$, there exists constants ε_j and μ_j such that

$$\begin{aligned}&\varepsilon(x_1, x_3) = \varepsilon_1, \quad \mu(x_1, x_3) = \mu_1, \quad \beta(x_1, x_3) = 0 \quad \text{for} \quad x_3 \geq b_1,\\ &\varepsilon(x_1, x_3) = \varepsilon_2, \quad \mu(x_1, x_3) = \mu_2, \quad \beta(x_1, x_3) = 0 \quad \text{for} \quad x_3 \leq b_2,\end{aligned}$$

where ε_j, μ_j and b_j are positive constants, $j = 1, 2$. As in [36], the following assumptions need to be satisfied:

(1) $\varepsilon(x), \mu(x)$ and $\beta(x)$ are all real valued L^∞ functions, $\varepsilon(x) \geq \varepsilon_0, \mu(x) \geq \mu_0$ and $\beta(x) \geq 0$, where ε_0 and μ_0 are positive constants;

(2) $d = 1 - k\beta \geq d_0 > 0$, for some positive constant d_0.

We note that the first assumption is a technical one and the second assumption is a essential one needed for the following numerical analysis. In fact, the second assumption is relatively reasonable since β is generally small.

Next some notation is introduced for proposing the weak formulation of the problem. Let $(\mathbf{E}_\mathrm{I}, \mathbf{H}_\mathrm{I})$ be the incoming plane waves that are incident upon the grating surface from the top

$$\mathbf{E}_\mathrm{I} = \tilde{s}e^{i\tilde{q}.x}, \quad \mathbf{H}_\mathrm{I} = \tilde{p}e^{i\tilde{q}.x}, \quad \tilde{s} = \frac{\tilde{p}\times\tilde{q}}{\omega\varepsilon_1}, \quad \tilde{q}\cdot\tilde{q} = \omega^2\varepsilon_1\mu_1, \quad \tilde{p}\cdot\tilde{q} = 0,$$

where the incident wave vector $\tilde{q}$ takes the form

$$\tilde{q} = (\alpha, 0, -\beta_1)^\mathrm{T} = \omega\sqrt{\varepsilon_1\mu_1}(sin\theta, 0, -cos\theta)^\mathrm{T},$$

and $0 \leq \theta < \pi$ is the incidence angle. The grating diffraction theory motivates to look for quasi-periodic solutions, i.e. solutions $(\mathbf{E}, \mathbf{H})$ such that $(E_\alpha, H_\alpha) = e^{-i\alpha x_1}(\mathbf{E}, \mathbf{H})$ are L-periodic in x_1. Under the radiation condition imposed on the scattering problem, it is known that the electromagnetic fields ($\mathbf{E}$,$\mathbf{H}$) is composed of bounded outgoing plane wave, plus the incident wave ($\mathbf{E}_\mathrm{I}$,$\mathbf{H}_\mathrm{I}$) above the structure.

2.2. The DtN formulation

Thanks to the periodic structure, we restrict our discussion to the bounded domain (see Fig. 2.1)

$$\Omega = \{(x_1, x_3) : 0 < x_1 < L,\ b_2 < x_3 < b_1\},$$

along with the artificial boundaries $\Gamma_j = \{(x_1, x_3) : 0 < x_1 < L, x_3 = b_j\}, j = 1, 2$.

For any quasi-periodic function $f \in H^{1/2}(\Gamma_j), T^{(j)}$ is the Dirichlet-to-Neumann(DtN) operator defined by

$$T^{(j)} f(x_1) = \sum_{n \in Z} i\beta_j^n f^{(n)} e^{i(\alpha_n+\alpha)x_1}, \quad 0 < x_1 < L, \quad j = 1, 2, \tag{2.4}$$

where $\alpha_n = 2\pi n/L, \beta_1^0 = \beta$ and the Fourier coefficient $f^{(n)}$ and the coefficient β_j^n are respectively given by

$$f^{(n)} = \frac{1}{L} \int_0^L f(x) e^{-i(\alpha_n+\alpha)x_1} dx_1, \tag{2.5}$$

$$\beta_j^n = \begin{cases} \left(k_j^2 - (\alpha_n+\alpha)^2\right)^{\frac{1}{2}}, & \text{if} \quad k_j^2 \geq (\alpha_n+\alpha)^2, \\ i\left((\alpha_n+\alpha)^2 - k_j^2\right)^{\frac{1}{2}}, & \text{if} \quad k_j^2 < (\alpha_n+\alpha)^2. \end{cases} \tag{2.6}$$

Let $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, e)^{\mathrm{T}}$, $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, h)^{\mathrm{T}}$. One can straightforwardly verify that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{h}_1$ and $\mathbf{h}_2$ can be expressed in terms of e and h. Then we obtain two coupled equations for e and h. Following the procedure described in [36], the scattering problem is finally reduced to solving the following problem:

$$\begin{aligned}
&-\nabla \cdot \left(\frac{1}{\mu}\nabla e\right) - \frac{\gamma^2}{\mu} e + i\omega\nabla \cdot (\beta\nabla h) - i\omega\gamma^2\beta h = 0 && \text{in } \Omega, \\
&-\nabla \cdot \left(\frac{1}{\varepsilon}\nabla h\right) - \frac{\gamma^2}{\varepsilon} h - i\omega\nabla \cdot (\beta\nabla e) + i\omega\gamma^2\beta e = 0 && \text{in } \Omega, \\
&\frac{\partial e}{\partial n} - T^{(1)} e = -2i\beta_1 \tilde{s}_3 e^{i\alpha x_1 - i\beta_1 b_1} && \text{on } \Gamma_1, \\
&\frac{\partial h}{\partial n} - T^{(1)} h = -2i\beta_1 \tilde{p}_3 e^{i\alpha x_1 - i\beta_1 b_1} && \text{on } \Gamma_1, \\
&\frac{\partial e}{\partial n} - T^{(2)} e = 0, \quad \frac{\partial h}{\partial n} - T^{(2)} h = 0 && \text{on } \Gamma_2.
\end{aligned} \tag{2.7}$$

Define the following space which includes all the quasi-periodic functions:

$$X(\Omega) = \left\{ f \in H^1(\Omega) \,:\, f(0, x_3) = e^{-i\alpha L} f(L, x_3) \right\}.$$

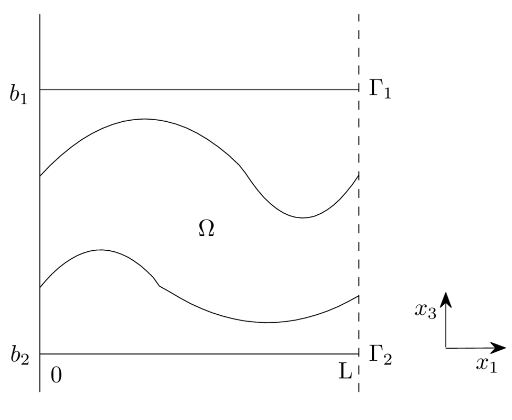

Fig. 2.1. The grating problem geometry in one period L.

Let $u=(e,h)^{\mathrm{T}}, v=(p,q)^{\mathrm{T}}$ and $w=(\varphi,\psi)^{\mathrm{T}}$. By integration by parts, we can easily derive the weak formulation of the scattering problem (2.7): Giving an incident plane wave

$$e_I=\tilde{s}_3 e^{i\alpha x_1-i\beta_1 x_3},\quad h_I=\tilde{p}_3 e^{i\alpha x_1-i\beta_1 x_3},$$

find $u\in(X(\Omega))^2$ such that

$$A(u,v)=\langle f_{\mathrm{I}},v\rangle,\quad \forall\, v\in\big(X(\Omega)\big)^2, \tag{2.8}$$

where the sesquilinear form A on $(X(\Omega))^2\times(X(\Omega))^2$ is defined as

$$\begin{aligned}A(u,v)=&\int_\Omega\left(\frac{1}{\mu}\nabla e\cdot\nabla\overline{p}-\frac{\gamma^2}{\mu}e\overline{p}-i\omega\beta\nabla h\cdot\nabla\overline{p}-i\omega\gamma^2\beta h\overline{p}\right)dx\\&+\int_\Omega\left(\frac{1}{\varepsilon}\nabla h\cdot\nabla\overline{q}-\frac{\gamma^2}{\varepsilon}h\overline{q}+i\omega\beta\nabla e\cdot\nabla\overline{q}+i\omega\gamma^2\beta e\overline{q}\right)dx\\&-\sum_{j=1}^{2}\frac{1}{\mu_j}\int_{\Gamma_j}(T^{(j)}e)\overline{p}dx_1-\sum_{j=1}^{2}\frac{1}{\varepsilon_j}\int_{\Gamma_j}(T^{(j)}h)\overline{q}dx_1,\end{aligned} \tag{2.9}$$

and

$$\langle f_{\mathrm{I}},v\rangle=-\frac{1}{\mu_1}\int_{\Gamma_1}2i\beta_1 e_I\overline{p}dx_1-\frac{1}{\varepsilon_1}\int_{\Gamma_1}2i\beta_1 h_I\overline{q}dx_1. \tag{2.10}$$

In [36], the uniqueness and existence of weak solutions to (2.8) was established for all but possibly a discrete set of frequencies ω. Here we simply assume that the variational problem (2.8) has unique solutions in $(X(\Omega))^2$. Then the following inf-sup condition:

$$\sup_{0\neq v\in(X(\Omega))^2}\frac{|A(w,v)|}{\|v\|_{(H^1(\Omega))^2}}\geq\gamma\|w\|_{(H^1(\Omega))^2},\quad\forall\, w\in\big(X(\Omega)\big)^2 \tag{2.11}$$

with a constant $\gamma>0$, based on the general theory of Babuška and Aziz [4], implies the estimate

$$\|u\|_{(H^1(\Omega))^2}\leq C_0\big(\|e_{\mathrm{I}}\|_{L^2(\Gamma_1)}+\|h_{\mathrm{I}}\|_{L^2(\Gamma_1)}\big). \tag{2.12}$$

2.3. The PML formulation

Now we turn to the description of absorbing PML layers. We surround the computational domain Ω with two PML layers Ω_j^{PML} of thickness $\delta_j, j=1,2$, where

$$\begin{aligned}\Omega_1^{\mathrm{PML}}&=\{(x_1,x_3)\,:\,0<x_1<L,\,b_1<x_3<b_1+\delta_1\},\\\Omega_2^{\mathrm{PML}}&=\{(x_1,x_3)\,:\,0<x_1<L,\,b_2-\delta_2<x_3<b_2\}.\end{aligned}$$

Let $s(x_3)=s_1(x_3)+\mathrm{i}s_2(x_3)$ be model medium property satisfying

$$s_1,\,s_2\in C(\mathbb{R}),\quad s_1\geq1,\quad s_2\geq0,\quad\text{and}\quad s(x_3)=1,\quad b_2\leq x_3\leq b_1. \tag{2.13}$$

As stated in [15], compared with the original PML condition which sets $s_1\equiv1$ in the PML domain, a variable s_1 chosen here can attenuate both the outgoing and evanescent waves. The advantage of this extension makes our following discussed method insensitive to the distance of the PML region from the structure.

According to the general idea developed for designing PML absorbing layers, we introduce the PML differential operators

$$\mathcal{L}_1 := \nabla \cdot \left(\frac{1}{\mu} \mathbb{A} \nabla \right) + \frac{\gamma^2}{\mu} s(x_3),$$
$$\mathcal{L}_2 := \nabla \cdot \left(\frac{1}{\varepsilon} \mathbb{A} \nabla \right) + \frac{\gamma^2}{\varepsilon} s(x_3),$$
$$\mathcal{L}_3 := -i\omega \nabla \cdot (\beta \nabla) + i\omega \gamma^2 \beta,$$

where

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & 0 \\ 0 & \mathbb{A}_{22} \end{pmatrix} = \begin{pmatrix} s(x_3) & 0 \\ 0 & \dfrac{1}{s(x_3)} \end{pmatrix}.$$

The PML equations in the PML region (cf. [36]) can be written as

$$\mathcal{L}_1(\widehat{e} - e_{\mathrm{I}}) = 0 \quad \text{in } \Omega_1^{\mathrm{PML}}, \quad \mathcal{L}_1(\widehat{e}) = 0 \quad \text{in } \Omega_2^{\mathrm{PML}},$$
$$\mathcal{L}_2(\widehat{h} - h_{\mathrm{I}}) = 0 \quad \text{in } \Omega_1^{\mathrm{PML}}, \quad \mathcal{L}_2(\widehat{h}) = 0 \quad \text{in } \Omega_2^{\mathrm{PML}}.$$

Introduce the differential operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_3 \\ -\mathcal{L}_3 & \mathcal{L}_2 \end{pmatrix},$$

and $D = \{(x_1, x_3) : 0 < x_1 < L,\ b_2 - \delta_2 < x_3 < b_1 + \delta_1\}$. By using the assumption (2.13), we can formulate the desired PML model to be numerically solved in this paper

$$\mathcal{L}\widehat{u} = -g \quad \text{in } D \tag{2.14}$$

with a quasi-periodic boundary condition in x_1 direction

$$\widehat{u}(0, x_3) = e^{-\mathrm{i}\alpha L} \widehat{u}(L, x_3), \quad b_2 - \delta_2 \le x_3 \le b_1 + \delta_1,$$

and the Dirichlet condition

$$\widehat{u} = u_{\mathrm{I}} \quad \text{on } \Gamma_1^{\mathrm{PML}} = \{(x_1, x_3) : 0 < x_1 < L,\ x_3 = b_1 + \delta_1\},$$
$$\widehat{u} = 0 \quad \text{on } \Gamma_2^{\mathrm{PML}} = \{(x_1, x_3) : 0 < x_1 < L,\ x_3 = b_2 - \delta_2\},$$

where $\widehat{u} = (\widehat{e}, \widehat{h})^{\mathrm{T}}$, $u_{\mathrm{I}} = (e_{\mathrm{I}}, h_{\mathrm{I}})^{\mathrm{T}}$ and the function g is defined as

$$g = \begin{cases} -\mathcal{L} u_{\mathrm{I}} & \text{in } \Omega_1^{\mathrm{PML}}, \\ 0, & \text{otherwise.} \end{cases}$$

Introduce the spaces

$$X(G) = \left\{ f \in H^1(G) : f(0, x_3) = e^{-i\alpha L} f(L, x_3) \right\}, \quad \forall G \subset D,$$
$$X^0(D) = \left\{ f \in X(D) : f = 0 \text{ on } \Gamma_1^{\mathrm{PML}} \bigcup \Gamma_2^{\mathrm{PML}} \right\}.$$

Now we consider the weak formulation of (2.14) which reads: Find $\widehat{u} \in (X(D))^2$ such that $\widehat{u} = u_{\mathrm{I}}$ on Γ_1^{PML}, $\widehat{u} = 0$ on Γ_2^{PML}, and

$$A_D(\widehat{u}, v) = \int_D g \cdot \overline{v} dx, \quad \forall v \in \left(X^0(D) \right)^2, \tag{2.15}$$

where for $G \subseteq D$, the sesquilinear form A_G on $(X(G))^2 \times (X(G))^2$ is defined as

$$\begin{aligned} A_G(\widehat{u}, v) = &\int_G \left(\frac{1}{\mu} \mathbb{A} \nabla \widehat{e} \cdot \nabla \overline{p} - \frac{\gamma^2}{\mu} s(x_3) \widehat{e} \overline{p} - i\omega\beta \nabla \widehat{h} \cdot \nabla \overline{p} - i\omega\gamma^2 \beta \widehat{h} \overline{p} \right) dx \\ &+ \int_G \left(\frac{1}{\varepsilon} \mathbb{A} \nabla \widehat{h} \cdot \nabla \overline{q} - \frac{\gamma^2}{\varepsilon} s(x_3) \widehat{h} \overline{q} + i\omega\beta \nabla \widehat{e} \cdot \nabla \overline{q} + i\omega\gamma^2 \beta \widehat{e} \overline{q} \right) dx. \end{aligned} \tag{2.16}$$

For the PML model, we introduce the following DtN operator $T^{(j,\mathrm{PML})}$ in [15]:

$$T^{(j,\mathrm{PML})} f(x_1) = \sum_{n \in \mathbb{Z}} i\beta_j^n \coth\big(- i\beta_j^n \sigma_j \big) f^{(n)} e^{i(\alpha_n + \alpha)x_1}, \quad 0 < x_1 < L, \quad j = 1, 2$$

for any quasi-periodic function f, where $\coth(\nu) = (e^{\nu} + e^{-\nu})/(e^{\nu} - e^{-\nu})$ and

$$\sigma_1 = \int_{b_1}^{b_1 + \delta_1} s(\nu) d\nu, \quad \sigma_2 = \int_{b_2 - \delta_2}^{b_2} s(\nu) d\nu. \tag{2.17}$$

Similar to the argument in [36], we arrive at the equivalent formulation of (2.15) in the domain Ω: Find $\widehat{u} \in (X(\Omega))^2$ such that

$$A^{\mathrm{PML}}(\widehat{u}, v) = \langle \widetilde{f}_{\mathrm{I}}, v \rangle, \quad \forall\, v \in \big(X(\Omega) \big)^2, \tag{2.18}$$

where the sesquilinear form A on $(X(\Omega))^2 \times (X(\Omega))^2$ is defined as

$$\begin{aligned} A^{\mathrm{PML}}(\widehat{u}, v) = &\int_\Omega \left(\frac{1}{\mu} \nabla \widehat{e} \cdot \nabla \overline{p} - \frac{\gamma^2}{\mu} \widehat{e} \overline{p} - i\omega\beta \nabla \widehat{h} \cdot \nabla \overline{p} - i\omega\gamma^2 \beta \widehat{h} \overline{p} \right) dx \\ &+ \int_\Omega \left(\frac{1}{\varepsilon} \nabla \widehat{h} \cdot \nabla \overline{q} - \frac{\gamma^2}{\varepsilon} \widehat{h} \overline{q} + i\omega\beta \nabla \widehat{e} \cdot \nabla \overline{q} + i\omega\gamma^2 \beta \widehat{e} \overline{q} \right) dx \\ &- \sum_{j=1}^{2} \frac{1}{\mu_j} \int_{\Gamma_j} \big(T^{(j,\mathrm{PML})} \widehat{e} \big) \overline{p} dx_1 - \sum_{j=1}^{2} \frac{1}{\varepsilon_j} \int_{\Gamma_j} \big(T^{(j,\mathrm{PML})} \widehat{h} \big) \overline{q} dx_1, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \langle \widetilde{f}_{\mathrm{I}}, v \rangle = &- \frac{1}{\mu_1} \int_{\Gamma_1} i\beta_1 \big(1 + \coth(-i\beta_1 \sigma_1) \big) e_I \overline{p} dx_1 \\ &- \frac{1}{\varepsilon_1} \int_{\Gamma_1} i\beta_1 \big(1 + \coth(-i\beta_1 \sigma_1) \big) h_I \overline{q} dx_1. \end{aligned}$$

We remark that the relation of the two variational equations (2.15) and (2.18) and the well-posedness of their solutions are studied in [36]. Throughout this paper, we assume that the variational problem (2.15) has a unique solution.

3. The Discrete Problem

In this section, we develop the PML finite element approximation and the PML finite volume approximation of the PML problem (2.15). In addition, we will give the a posteriori error estimate which plays an important role for the adaptive PML finite volume method.

3.1. Finite element approximation

Let $\mathcal{M}_h$ be a regular triangulation of the domain D. It is required that any triangle $T \in \mathcal{M}_h$ must be completely included in $\Omega, \Omega_1^{\mathrm{PML}}$, or Ω_2^{PML}. To deal with the quasi-periodic boundary conditions, we further assume that if $(0, z)$ is a node on the left boundary, then (L, z) must be a node on the right boundary, and vice versa. Let U_h be space of the conforming linear finite element over $\mathcal{M}_h$, i.e.

$$\begin{aligned} U_h := \big\{ &\varphi_h \in X(D) : \varphi_h|_T \in P_1(T), \forall\, T \in \mathcal{M}_h, \\ &\varphi_h(0, x_3) = e^{-i\alpha L}\varphi_h(L, x_3),\, b_2 - \delta_2 < x_3 < b_1 + \delta_1 \big\}, \end{aligned}$$

where $P_1(T)$ is the space of all piecewise linear polynomials. Denote by $U_h^0 = U_h \cap X^0(D)$, and the operator I_h: $(C(D))^2 \to (U_h)^2$ is chosen as the standard finite element interpolation operator.

The PML finite element approximation to (2.15) reads as follows: Find $\widehat{u}_h = (\widehat{e}_h, \widehat{h}_h)^{\mathrm{T}} \in (U_h)^2$ such that $\widehat{u}_h = I_h u_{\mathrm{I}}$ on Γ_1^{PML}, $\widehat{u}_h = 0$ on Γ_2^{PML}, and

$$A_D(\widehat{u}_h, v_h) = \int_D g \cdot \overline{v_h} dx, \quad \forall\, v_h \in \big(U_h^0\big)^2. \tag{3.1}$$

In this work, the existence and uniqueness of the discrete problem (3.1) for sufficiently small h can be proved by using the inf-sup condition satisfied by the continuous problem (2.15), the argument of Schatz [29] and the general theory in [4]. Throughout this paper, we assume that the discrete problem (3.1) has a unique solution.

3.2. Finite volume approximation

Let $\mathcal{M}_h^*$ be a dual partition in D related to $\mathcal{M}_h$, and its elements are closed polygons called the control volumes. Let T_Q be the barycenter of the element $T \in \mathcal{M}_h$. We connect T_Q with line segments to the midpoints of the edges of T and divide the element T into three quadrilaterals T_P, where $P \in \mathcal{N}(K)$ and $\mathcal{N}(K)$ is the set of vertices of T. For each node P of $\mathcal{M}_h$, the corresponding control volume T_P^* is constructed by the union of the subregions T_P sharing the node P. In the same way as [9], we finally obtain a collection of control volumes covering the domain D, which is called the dual partition $\mathcal{M}_h^*$. The test function space V_h corresponding to $\mathcal{M}_h^*$ is taken as the piecewise constant function space, i.e.

$$\begin{aligned} V_h := \big\{ &\psi_h \in L^2(\overline{D}) : \psi_h|_{T_P^*} \text{ is constant for } P \in \mathcal{N}^I, \\ &\text{and } \psi_h|_{T_P^*} = 0 \text{ for } P \in \mathcal{N}^j,\ j = 1, 2 \big\}, \end{aligned}$$

where $\mathcal{N}^I$ and $\mathcal{N}^j$ respectively denote the sets of all vertices on the interior domain of D and the quasic-periodic boundaries and the sets of all vertices on $\Gamma_j^{\mathrm{PML}}, j = 1, 2$. For $w_h \in (U_h)^2$, let $\mathcal{I}_h^*$ be the interpolation projection operator of w_h onto the test space $(V_h)^2$

$$\mathcal{I}_h^* w_h = \sum_{P \in \mathcal{N}^I} w_h(P) \chi_P(x),$$

where χ_P is the characteristic function of the control volume T_P^*.

Now we formulate the finite volume scheme for the PML problem (2.14). The corresponding variational formulation is derived by multiplying (2.14) by $\mathcal{I}_h^* v_h$, integrating by parts over each T_P^* and summing over all $P \in \mathcal{N}^I$

$$\widetilde{A}_D\big(\widehat{u}, \mathcal{I}_h^* v_h\big) = -\int_D g \cdot \mathcal{I}_h^* \overline{v_h} dx, \quad \forall\, v_h \in (U_h)^2, \tag{3.2}$$

where $v_h=(p_h,q_h)^{\mathrm{T}}$, n is the outer-normal vector of the associated domain, and

$$\begin{aligned}\widetilde{A}_D\big(\widehat{u},\mathcal{I}_h^*v_h\big)=&\sum_{P\in\mathcal{N}^I}\overline{p}_h(P)\bigg(\int_{\partial T_P^*}\Big(\frac{1}{\mu}\mathbb{A}\nabla\widehat{e}\cdot n-i\omega\beta\nabla\widehat{h}\cdot n\Big)ds\\&\qquad+\int_{T_P^*}\Big(\frac{\gamma^2}{\mu}s(x_3)\widehat{e}+i\omega\gamma^2\beta\widehat{h}\Big)dx\bigg)\\&+\sum_{P\in\mathcal{N}^I}\overline{q}_h(P)\bigg(\int_{\partial T_P^*}\Big(\frac{1}{\varepsilon}\mathbb{A}\nabla\widehat{h}\cdot n+i\omega\beta\nabla\widehat{e}\cdot n\Big)ds\\&\qquad+\int_{T_P^*}\Big(\frac{\gamma^2}{\varepsilon}s(x_3)\widehat{h}-i\omega\gamma^2\beta\widehat{e}\Big)dx\bigg).\end{aligned}$$

The PML finite volume approximation to (3.2) reads as follows: Find $\widetilde{u}_h=(\widetilde{e}_h,\widetilde{h}_h)^{\mathrm{T}}\in(U_h)^2$ such that $\widetilde{u}_h=I_hu_{\mathrm{I}}$ on Γ_1^{PML}, $\widetilde{u}_h=0$ on Γ_2^{PML}, and

$$\widetilde{A}_D\big(\widetilde{u}_h,\mathcal{I}_h^*v_h\big)=-\int_D g\cdot\mathcal{I}_h^*\overline{v_h}dx,\quad\forall\,v_h\in(U_h)^2.\tag{3.3}$$

We must point out that, to the best of our knowledge, no relevant results on the well-posedness and the priori error estimate have been derived for (3.3). Here our interest is focused on the a posteriori and convergence analysis for the adaptive PML finite volume method. Thus, in the following it is assumed that the discrete problem (3.3) has a unique solution.

3.3. The a posteriori error estimate

We begin with introducing some notation to define the error indicators. For any $T\in\mathcal{M}_h$, denote by h_T its diameter, we define the element residual

$$\mathbf{R}_T:=\mathcal{L}\widetilde{u}_h|_T+g|_T.$$

Let $\mathcal{B}_h$ be the set of all the sides that do not lie on Γ_1^{PML} and Γ_2^{PML}. For any $F\in\mathcal{B}_h$, h_F stands for its length. Given an interior edge $F\in\mathcal{B}_h$ which is the common edge of T_1 and $T_2\in\mathcal{M}_h$, we define the jump residual across F as

$$\begin{aligned}\mathbf{J}_F^{(1)}&=\Big(\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h-i\omega\beta\nabla\widetilde{h}_h\Big)\Big|_{T_1}\cdot n_1+\Big(\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h-i\omega\beta\nabla\widetilde{h}_h\Big)\Big|_{T_2}\cdot n_2,\\\mathbf{J}_F^{(2)}&=\Big(\frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h+i\omega\beta\nabla\widetilde{e}_h\Big)\Big|_{T_1}\cdot n_1+\Big(\frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h+i\omega\beta\nabla\widetilde{e}_h\Big)\Big|_{T_2}\cdot n_2,\end{aligned}\tag{3.4}$$

where n_j denotes the unit outward normal vector to the boundary of $T_j, j=1,2$. Define

$$\begin{aligned}\Gamma_1&=\{(x_1,x_3):x_1=0,\,b_2-\delta_2<x_3<b_1+\delta_1\},\\\Gamma_2&=\{(x_1,x_3):x_1=L,\,b_2-\delta_2<x_3<b_1+\delta_1\}.\end{aligned}$$

If $F=\Gamma_1\bigcap\partial T$ for some element $T\in\mathcal{M}_h$, F' is a corresponding edge on Γ_2 which also belongs to some element T', the jump residuals across F and F' are defined by

$$\mathbf{J}_F^{(1)}=\Big(\frac{1}{\mu}\mathbb{A}_{11}\frac{\partial\widetilde{e}_h}{\partial x_1}-i\omega\beta\frac{\partial\widetilde{h}_h}{\partial x_1}\Big)\Big|_{T_1}-e^{-i\alpha L}\Big(\frac{1}{\mu}\mathbb{A}_{11}\frac{\partial\widetilde{e}_h}{\partial x_1}-i\omega\beta\frac{\partial\widetilde{h}_h}{\partial x_1}\Big)\Big|_{T_1'},\tag{3.5a}$$

$$\mathbf{J}_F^{(2)} = \left(\frac{1}{\varepsilon}\mathbb{A}_{11}\frac{\partial \widetilde{h}_h}{\partial x_1} + i\omega\beta\frac{\partial \widetilde{e}_h}{\partial x_1}\right)\Big|_{T_1} - e^{-i\alpha L}\left(\frac{1}{\varepsilon}\mathbb{A}_{11}\frac{\partial \widetilde{h}_h}{\partial x_1} + i\omega\beta\frac{\partial \widetilde{e}_h}{\partial x_1}\right)\Big|_{T_1'}, \tag{3.5b}$$

$$\mathbf{J}_{F'}^{(1)} = e^{i\alpha L}\left(\frac{1}{\mu}\mathbb{A}_{11}\frac{\partial \widetilde{e}_h}{\partial x_1} - i\omega\beta\frac{\partial \widetilde{h}_h}{\partial x_1}\right)\Big|_{T_1} - \left(\frac{1}{\mu}\mathbb{A}_{11}\frac{\partial \widetilde{e}_h}{\partial x_1} - i\omega\beta\frac{\partial \widetilde{h}_h}{\partial x_1}\right)\Big|_{T_1'}, \tag{3.5c}$$

$$\mathbf{J}_{F'}^{(2)} = e^{i\alpha L}\left(\frac{1}{\varepsilon}\mathbb{A}_{11}\frac{\partial \widetilde{h}_h}{\partial x_1} + i\omega\beta\frac{\partial \widetilde{e}_h}{\partial x_1}\right)\Big|_{T_1} - \left(\frac{1}{\varepsilon}\mathbb{A}_{11}\frac{\partial \widetilde{h}_h}{\partial x_1} + i\omega\beta\frac{\partial \widetilde{e}_h}{\partial x_1}\right)\Big|_{T_1'}. \tag{3.5d}$$

The local error indicator η_T for any $T \in \mathcal{M}_h$ is defined as follows:

$$\eta_T = \max_{x\in\widetilde{T}} \mathbf{w}(x_3)\left\{ h_T\|\mathbf{R}_T\|_{(L^2(T))^2} + \left(\frac{1}{2}\sum_{F\subset\partial T} h_F\|\mathbf{J}_F\|^2_{(L^2(F))^2}\right)^{\frac{1}{2}}\right\}, \tag{3.6}$$

where $\mathbf{J}_F = (\mathbf{J}_F^{(1)}, \mathbf{J}_F^{(2)})^{\mathrm{T}}, \widetilde{T}$ is the union of all the elements in $\mathcal{M}_h$ with nonempty intersection with T, and

$$\mathbf{w}(x_3) = \begin{cases} 1, & \text{if} \quad x \in \Omega, \\ |s(x_3)|e^{-R_j(x_3)}, & \text{if} \quad x \in \overline{\Omega_j^{\mathrm{PML}}} \end{cases}$$

with $R_j(x_3)$ being defined below, $j = 1, 2$.

We now state the main result, which lays a theoretical foundation for the following mesh adaptive strategy.

Theorem 3.1. *Assume that u and $\widetilde{u}_h$ are the solutions of* (2.8) *and* (3.3), *respectively. Then there exists a positive constant C, depending only on the minimum angle of the mesh $\mathcal{M}_h$ such that the following a posteriori error estimate holds:*

$$\begin{aligned}\|u - \widetilde{u}_h\|_{(H^1(\Omega))^2} \le C\Bigg((1 + C_1 + C_2)\Bigg(\sum_{T\in\mathcal{M}_h}\eta_T^2\Bigg)^{\frac{1}{2}} + \widehat{C}\mathbf{M}_1\|\widetilde{u}_h - u_I\|_{(L^2(\Gamma_1))^2} \\ + \widehat{C}\mathbf{M}_2\|\widetilde{u}_h\|_{(L^2(\Gamma_2))^2} + \widehat{C}\mathbf{M}_3\|u_I - I_h u_I\|_{(L^2(\Gamma_1^{PML}))^2}\Bigg),\end{aligned}$$

where the constants $\mathbf{M}_j, \mathbf{M}_3, C_j$ and $\widehat{C}$ are defined in the following Lemmas 4.4, 4.9, *and* 4.10, *respectively, $j = 1, 2$.*

It must be said that the error estimate can be viewed as a extension of the related work for solving the grating problem(cf. [15]). As the corresponding constants used in [15], $\mathbf{M}_j$, $j = 1, 2$, and $\mathbf{M}_3$ decays exponentially with the PML parameters σ_j^R and σ_j^I, where σ_j^R and σ_j^I is respectively the real and imaginary parts of σ_j defined in (2.17). In particular, the exponential decay factors $e^{-R_j(x_3)}$ in the PML region Ω_j^{PML} allows us to take thicker PML layers, and the result is that the thicker PML layers allow a smaller PML medium property so as to ensure numerical stability.

4. The a Posteriori Error Analysis

In the section, we derive the a posteriori error estimate in Theorem 3.1 and the local lower bound in Theorem 4.1. We start with the following lemma (cf. [15]).

Lemma 4.1. *For any $f \in X(\Omega)$, there holds*

$$\|f\|_{H^{\frac{1}{2}}(\Gamma_j)} \leq \widehat{C}\|f\|_{H^1(\Omega)}$$

with $\widehat{C} = \sqrt{1+(b_1-b_2)^{-1}}$. *Here if* $f(x_1, b_j) = \sum_{n\in Z} f^{(n)}(b_j)e^{i(\alpha_n+\alpha)x_1}$ *on* Γ_j,

$$\|f\|_{H^{\frac{1}{2}}(\Gamma_j)} = \left(L\sum_{n\in Z}\left(1+|\alpha_n+\alpha|^2\right)^{\frac{1}{2}}|f^{(n)}(b_j)|^2\right)^{\frac{1}{2}}, \quad j=1,2.$$

Let $\mathcal{I}_h : X(D) \to U_h$ be the Scott-Zhang interpolation operator. Both the operators $\mathcal{I}_h$ and $\mathcal{I}_h^*$ keep the quasi-periodic boundary conditions and enjoy the following properties (see, e.g. [30,33]).

Lemma 4.2. *For any $f \in X(D)$, there exists a function $f_h = \mathcal{I}_h f \in U_h$ such that*

$$\begin{aligned}
&\|f-f_h\|_{L^2(T)} \leq Ch_T\|\nabla f\|_{L^2(\widetilde{T})},\\
&\|f-f_h\|_{L^2(F)} \leq Ch_F^{\frac{1}{2}}\|\nabla f\|_{L^2(\widetilde{F})},\\
&\|f_h-\mathcal{I}_h^* f_h\|_{L^2(T)} \leq Ch_T\|\nabla f_h\|_{L^2(T)},\\
&\|\nabla f_h\|_{L^2(T)} \leq C\|\nabla f\|_{L^2(\widetilde{T})},
\end{aligned}$$

where $\widetilde{T}$ and $\widetilde{F}$ are the union of all the elements in $\mathcal{M}_h$ with nonempty intersection with T and F, respectively.

4.1. Error representation formulae

For any $v \in (X(\Omega))^2$, we extend v to $\Omega_j^{\rm PML}$ as follows:

$$\widetilde{v} = \sum_{n\in Z}\frac{\overline{\zeta}_j^n(x_3)}{\overline{\zeta}_j^n(b_j)}v^{(n)}e^{i(\alpha_n+\alpha)x_1} \quad \text{in } \Omega_j^{\rm PML}, \quad j=1,2,$$

where the definition of $\zeta_j^n(x_3)$ can be found in [15], $v^{(n)}$ is the Fourier coefficient of the vector function as defined above, and

$$v(x_1, b_j) = \sum_{n\in Z} v^{(n)}e^{i(\alpha_n+\alpha)x_1}.$$

It can be readily seen that $\widetilde{v} = v$ on Γ_j and $\mathcal{L}\overline{\widetilde{v}} = 0$ in $\Omega_j^{\rm PML}, j = 1,2$.

The following lemma can be proved in a fashion similar to that of [15, Lemma 4.1].

Lemma 4.3. *For any $v, w \in (X(\Omega))^2$, there holds*

$$\begin{aligned}
\int_{\Gamma_j} T^{(j,PML)}\varphi\overline{p}dx_1 &= -\int_{\Gamma_j}\varphi\frac{\partial\overline{\widetilde{p}}}{\partial n_j}dx_1,\\
\int_{\Gamma_j} T^{(j,PML)}\psi\overline{q}dx_1 &= -\int_{\Gamma_j}\psi\frac{\partial\overline{\widetilde{q}}}{\partial n_j}dx_1,
\end{aligned}$$

where n_j is unit outer normal to $\Omega_j^{PML}, j=1,2$.

Whenever there is no confusion, $\widetilde{v}$ is written as v in $\Omega_j^{\rm PML}$, $j=1,2$. Besides, we use the following notations:

$$\triangle_j^n = \left|k_j^2-(\alpha_n+\alpha)^2\right|^{\frac{1}{2}},\quad \mathcal{U}_j=\left\{n: k_j^2>(\alpha_n+\alpha)^2\right\},$$
$$\triangle_j^- = \min\left\{\triangle_j^n: n\in\mathcal{U}_j\right\},\quad \triangle_j^+ = \min\left\{\triangle_j^n: n\notin\mathcal{U}_j\right\},\quad j=1,2.$$

The following lemmas(cf. [15]) is important in the subsequent analysis.

Lemma 4.4. *For any $\varphi,\psi\in X(\Omega)$, there holds*

$$\int_{\Gamma_j}\big(T^{(j)}-T^{(j,PML)}\big)\varphi\overline{\psi}dx_1\le \mathbf{M}_j\|\varphi\|_{L^2(\Gamma_j)}\|\psi\|_{L^2(\Gamma_j)},$$

where

$$\mathbf{M}_j=\max\left(\frac{2\,\triangle_j^-}{e^{2\sigma_j^I\triangle_j^-}-1},\frac{2\,\triangle_j^+}{e^{2\sigma_j^R\triangle_j^+}-1}\right)$$

and σ_j^R,σ_j^I are the real and imaginary parts of σ_j defined in (2.17), *$j=1,2$.*

Lemma 4.5. *For any $v\in(X(\Omega))^2$, which is extended to be a vector function in $(X(D))^2$ as shown above, there holds*

$$\begin{aligned}A(u-\widetilde{u}_h,v)=&\int_D g\overline{v}dx-A_D(\widetilde{u}_h,v)+\frac{1}{\mu_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,PML)}\big)(\widetilde{e}_h-e_I)\overline{p}dx_1\\&+\frac{1}{\mu_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,PML)}\big)\widetilde{e}_h\overline{p}dx_1+\frac{1}{\varepsilon_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,PML)}\big)(\widetilde{h}_h-h_I)\overline{q}dx_1\\&+\frac{1}{\varepsilon_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,PML)}\big)\widetilde{h}_h\overline{q}dx_1-\frac{1}{\mu_1}\int_{\Gamma_1^{PML}}(e_I-I_he_I)\frac{1}{s(x_3)}\frac{\partial\overline{p}}{\partial x_3}dx_1\\&-\frac{1}{\varepsilon_1}\int_{\Gamma_1^{PML}}(h_I-I_hh_I)\frac{1}{s(x_3)}\frac{\partial\overline{q}}{\partial x_3}dx_1.\end{aligned}\tag{4.1}$$

Proof. From (2.8), (2.18) and the definition of the sesquilinear form A and $A^{\rm PML}$, we know that

$$\begin{aligned}A(u-\widehat{u},v)&=\langle f_{\rm I},v\rangle-\langle\widetilde{f}_{\rm I},v\rangle+A^{\rm PML}(\widehat{u},v)-A(\widehat{u},v)\\&=\frac{1}{\mu_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,\rm PML)}\big)(\widehat{e}-e_{\rm I})\overline{p}dx_1+\frac{1}{\mu_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,\rm PML)}\big)\widehat{e}\overline{p}dx_1\\&\quad+\frac{1}{\varepsilon_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,\rm PML)}\big)(\widehat{h}-h_{\rm I})\overline{q}dx_1+\frac{1}{\varepsilon_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,\rm PML)}\big)\widehat{h}\overline{q}dx_1,\end{aligned}\tag{4.2}$$

where we have used

$$\frac{\partial u_{\rm I}}{\partial n}-T^{(1)}u_{\rm I}=-2i\beta_1u_{\rm I},$$
$$\frac{\partial u_{\rm I}}{\partial n}-T^{(1,\rm PML)}u_{\rm I}=-i\beta_1\big(1+\coth(-i\beta_1\sigma_1)\big)u_{\rm I}$$

on Γ_1 to derive the last equality. It follows from (4.2) that

$$A(u-\widetilde{u}_h,v)=A(u-\widehat{u},v)+A^{\rm PML}(\widehat{u}-\widetilde{u}_h,v)$$

$$
\begin{aligned}
&-\sum_{j=1}^{2}\frac{1}{\mu_j}\int_{\Gamma_j}\big(T^{(j)}-T^{(j,\mathrm{PML})}\big)(\widehat{e}-\widetilde{e}_h)\overline{p}dx_1\\
&-\sum_{j=1}^{2}\frac{1}{\varepsilon_j}\int_{\Gamma_j}\big(T^{(j)}-T^{(j,\mathrm{PML})}\big)(\widehat{h}-\widetilde{h}_h)\overline{q}dx_1\\
=&\,A^{\mathrm{PML}}(\widehat{u}-\widetilde{u}_h,v)+\frac{1}{\mu_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,\mathrm{PML})}\big)(\widetilde{e}_h-e_{\mathrm{I}})\overline{p}dx_1\\
&+\frac{1}{\varepsilon_1}\int_{\Gamma_1}\big(T^{(1)}-T^{(1,\mathrm{PML})}\big)(\widetilde{h}_h-h_{\mathrm{I}})\overline{q}dx_1\\
&+\frac{1}{\mu_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,\mathrm{PML})}\big)\widetilde{e}_h\overline{p}dx_1+\frac{1}{\varepsilon_2}\int_{\Gamma_2}\big(T^{(2)}-T^{(2,\mathrm{PML})}\big)\widetilde{h}_h\overline{q}dx_1 \qquad (4.3)
\end{aligned}
$$

Using Lemma 4.3 yields

$$
\begin{aligned}
A^{\mathrm{PML}}(\widehat{u}-\widetilde{u}_h,v)=&\,A_\Omega(\widehat{u}-\widetilde{u}_h,v)-\sum_{j=1}^{2}\frac{1}{\mu_j}\int_{\Gamma_j}T^{(j,\mathrm{PML})}(\widehat{e}-\widetilde{e}_h)\overline{p}dx_1 \qquad (4.4)\\
&-\sum_{j=1}^{2}\frac{1}{\varepsilon_j}\int_{\Gamma_j}T^{(j,\mathrm{PML})}(\widehat{h}-\widetilde{h}_h)\overline{q}dx_1\\
=&\,A_\Omega(\widehat{u}-\widetilde{u}_h,v)+\sum_{j=1}^{2}\frac{1}{\mu_j}\int_{\Gamma_j}(\widehat{e}-\widetilde{e}_h)\frac{\partial\overline{p}}{\partial n}dx_1+\sum_{j=1}^{2}\frac{1}{\varepsilon_j}\int_{\Gamma_j}(\widehat{h}-\widetilde{h}_h)\frac{\partial\overline{q}}{\partial n}dx_1.
\end{aligned}
$$

From the Green formula and $\mathcal{L}\overline{v}=0$ in $\Omega_j^{\mathrm{PML}}, j=1,2$, we obtain that

$$
\begin{aligned}
A_{\Omega_j^{\mathrm{PML}}}(\widehat{u}-\widetilde{u}_h,v)=&\int_{\Omega_j^{\mathrm{PML}}}(\widehat{u}-\widetilde{u}_h)\cdot\mathcal{L}\overline{v}dx+\frac{1}{\mu_j}\int_{\partial\Omega_j^{\mathrm{PML}}}(\widehat{e}-\widetilde{e}_h)\mathbb{A}\nabla\overline{p}\cdot nds\\
&+\frac{1}{\varepsilon_j}\int_{\partial\Omega_j^{\mathrm{PML}}}\big(\widehat{h}-\widetilde{h}_h\big)\mathbb{A}\nabla\overline{q}\cdot nds\\
=&\,\frac{1}{\mu_j}\left(\int_{\Gamma_j}(\widehat{e}-\widetilde{e}_h)\frac{\partial\overline{p}}{\partial n}ds+\int_{\Gamma_j^{\mathrm{PML}}}(\widehat{e}-\widetilde{e}_h)\frac{1}{s(x_3)}\frac{\partial\overline{p}}{\partial n}ds\right)\\
&+\frac{1}{\varepsilon_j}\left(\int_{\Gamma_j}\big(\widehat{h}-\widetilde{h}_h\big)\frac{\partial\overline{q}}{\partial n}ds+\int_{\Gamma_j^{\mathrm{PML}}}\big(\widehat{h}-\widetilde{h}_h\big)\frac{1}{s(x_3)}\frac{\partial\overline{q}}{\partial n}ds\right),
\end{aligned}
$$

which together with (4.4) and (2.15) implies that

$$
\begin{aligned}
A^{\mathrm{PML}}(\widehat{u}-\widetilde{u}_h,v)=&\int_D g\overline{v}dx-A_D(\widetilde{u}_h,v)-\frac{1}{\mu_1}\int_{\Gamma_1^{\mathrm{PML}}}(\widehat{e}-\widetilde{e}_h)\frac{1}{s(x_3)}\frac{\partial\overline{p}}{\partial n}ds\\
&-\frac{1}{\varepsilon_1}\int_{\Gamma_1^{\mathrm{PML}}}\big(\widehat{h}-\widetilde{h}_h\big)\frac{1}{s(x_3)}\frac{\partial\overline{q}}{\partial n}ds,
\end{aligned}
$$

where we additionally employed $\widehat{u}=\widetilde{u}_h=0$ on Γ_2^{PML}. Then the proof is completed by plugging the above equality to (4.3). □

4.2. L^2-orthogonality and estimates for the residual

The following lemma will play a similar role as Galerkin orthogonality property for the classic finite element method and is important in deriving the a posteriori error analysis.

Lemma 4.6. *For the interior residual* $\mathbf{R}_T$ *and the jump residual* J_e*, there holds*

$$\sum_{T\in\mathcal{M}_h}\left(\int_T \mathbf{R}_T\cdot\mathcal{I}_h^*\overline{v}_h dx-\sum_{F\subset\partial T}\frac{1}{2}\int_F \mathbf{J}_F\cdot\mathcal{I}_h^*\overline{v}_h ds\right)=0,\quad \forall\, v_h\in(U_h)^2. \tag{4.5}$$

Proof. For each control volume $T_P^*\in\mathcal{M}_h^*$, the following vector equations are derived from (3.3) by taking $\mathcal{I}_h^*\overline{v_h}=(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$:

$$\int_{\partial T_P^*}\begin{pmatrix}\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h\cdot n-i\omega\beta\nabla\widetilde{h}_h\cdot n\\ \frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h\cdot n+i\omega\beta\nabla\widetilde{e}_h\cdot n\end{pmatrix}ds+\int_{T_P^*}\begin{pmatrix}\frac{\gamma^2}{\mu}s(x_3)\widetilde{e}_h+i\omega\gamma^2\beta\widetilde{h}_h\\ \frac{\gamma^2}{\varepsilon}s(x_3)\widetilde{h}_h-i\omega\gamma^2\beta\widetilde{e}_h\end{pmatrix}dx=-\int_{T_P^*}gdx.$$

Using the integration by parts, we easily obtain the following vector formulas:

$$\begin{aligned}&\int_{T_P^*}\begin{pmatrix}-\nabla\cdot\left(\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h\right)+i\omega\nabla\cdot\left(\beta\nabla\widetilde{h}_h\right)\\ -\nabla\cdot\left(\frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h\right)-i\omega\nabla\cdot\left(\beta\nabla\widetilde{e}_h\right)\end{pmatrix}dx\\
&=\sum_{T\in\mathcal{M}_h}\int_{\partial(T_P^*\cap T)}\begin{pmatrix}-\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h\cdot n+i\omega\beta\nabla\widetilde{h}_h\cdot n\\ -\frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h\cdot n-i\omega\beta\nabla\widetilde{e}_h\cdot n\end{pmatrix}ds\\
&=\int_{\partial T_P^*}\begin{pmatrix}-\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h\cdot n+i\omega\beta\nabla\widetilde{h}_h\cdot n\\ -\frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h\cdot n-i\omega\beta\nabla\widetilde{e}_h\cdot n\end{pmatrix}ds-\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}\frac{1}{2}\int_{F\cap T_P^*}\begin{pmatrix}\mathbf{J}_F^{(1)}\\ \mathbf{J}_F^{(2)}\end{pmatrix}ds.\end{aligned}$$

It follows from the above two vector equations that

$$\int_{T_P^*}(\mathcal{L}\widetilde{u}_h+g)dx-\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}\frac{1}{2}\int_{F\cap T_P^*}\mathbf{J}_F ds=0. \tag{4.6}$$

Then the proof is completed by doing dot product of (4.6) and $\mathcal{I}_h^*\overline{v}_h$, taking the sum over all $T_P^*\in\mathcal{M}_h^*$ and using the quasi-periodicity of the vector function v_h. $\square$

The two lemmas below establish robust estimates for the interior residual and jump residual.

Lemma 4.7. *There exists a constant* $C>0$ *that is independent of* h *such that the estimate*

$$\sum_{T\in\mathcal{M}_h}\int_T\mathbf{R}_T\cdot\left(\overline{v}-\mathcal{I}_h^*\overline{v}_h\right)dx\le C\sum_{T\in\mathcal{M}_h}h_T\|\mathbf{R}_T\|_{(L^2(T))^2}\|\nabla v\|_{(L^2(\widetilde{T}))^2}$$

holds true for the interior residual $\mathbf{R}_T$ *and* $v\in(X(D))^2$.

Proof. By triangle inequality and Lemma 4.2, we get

$$\begin{aligned}\|v-\mathcal{I}_h^*v_h\|_{(L^2(T))^2}&\le\|v-v_h\|_{(L^2(T))^2}+\|v_h-\mathcal{I}_h^*v_h\|_{(L^2(T))^2}\\
&\le\|p-p_h\|_{L^2(T)}+\|p_h-\mathcal{I}_h^*p_h\|_{L^2(T)}\\
&\quad+\|q-q_h\|_{L^2(T)}+\|q_h-\mathcal{I}_h^*q_h\|_{L^2(T)}\\
&\le Ch_T\left(\|\nabla p\|_{L^2(\widetilde{T})}+\|\nabla q\|_{L^2(\widetilde{T})}\right),\end{aligned}$$

where $v_h = \mathcal{I}_h v$. We have from the Cauchy-Schwarz inequality and the above inequality that

$$\sum_{T\in\mathcal{M}_h}\int_T \mathbf{R}_T\cdot\big(\overline{v}-\mathcal{I}_h^*\overline{v}_h\big)dx \le C\sum_{T\in\mathcal{M}_h} h_T\|\mathbf{R}_T\|_{(L^2(T))^2}\big(\|\nabla p\|_{L^2(\widetilde{T})}+\|\nabla q\|_{L^2(\widetilde{T})}\big),$$

which completes the proof. ☐

Lemma 4.8. *There exists a constant $C>0$ that is independent of h such that the estimate*

$$\begin{aligned}&\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}\frac{1}{2}\int_F \mathbf{J}_F\cdot\big(\overline{v}-\mathcal{I}_h^*\overline{v}_h\big)ds\\ &\le C\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T} h_F^{\frac{1}{2}}\|\mathbf{J}_F\|_{(L^2(F))^2}\|\nabla v\|_{(L^2(\widetilde{T}))^2}\end{aligned}$$

holds true for the jump residual $\mathbf{J}_F^{(j)}$ and $v\in(X(D))^2$, $j=1,2$.

Proof. First, it follows from Lemma 4.2 and the fact that $\mathcal{I}_h^* v_h$ denotes a piecewise constant vector function that

$$\begin{aligned}h_F^{-1}\|p-\mathcal{I}_h^*p_h\|^2_{L^2(F)} &\le C\left(h_F^{-2}\|p-\mathcal{I}_h^*p_h\|^2_{L^2(\widetilde{F})}+\|\nabla p\|^2_{L^2(\widetilde{F})}\right)\\ &\le C\left(h_F^{-2}\|p-p_h\|^2_{L^2(\widetilde{F})}+h_F^{-2}\|p_h-\mathcal{I}_h^*p_h\|^2_{L^2(\widetilde{F})}+\|\nabla p\|^2_{L^2(\widetilde{F})}\right)\\ &\le C\|\nabla p\|^2_{L^2(\widetilde{T})},\end{aligned}$$

where to derive the first inequality, we use the trace inequality

$$\|f\|^2_{L^2(F)}\le C\left(h_F^{-1}\|f\|^2_{L^2(T)}+h_F\|\nabla f\|^2_{L^2(T)}\right),\quad \forall\, f\in H^1(T),\quad F\subset\partial T,\quad T\in\mathcal{M}_h,$$

similarly, there holds

$$h_F^{-1}\|q-\mathcal{I}_h^*q_h\|^2_{L^2(F)}\le C\|\nabla q\|^2_{L^2(\widetilde{T})},$$

where $v_h=\mathcal{I}_h v$.

Then, using the above inequality and the Cauchy-Schwarz inequality leads to

$$\begin{aligned}&\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}\frac{1}{2}\int_F \mathbf{J}_F\cdot\big(\overline{v}-\mathcal{I}_h^*\overline{v}_h\big)ds\\ &=\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}\frac{1}{2}\int_F\left(\mathbf{J}_F^{(1)}\big(\overline{p}-\mathcal{I}_h^*\overline{p}_h\big)+\mathbf{J}_F^{(2)}\big(\overline{q}-\mathcal{I}_h^*\overline{q}_h\big)\right)ds\\ &\le C\sum_{T\in\mathcal{M}_h}\sum_{F\subset\partial T}h_F^{\frac{1}{2}}\left(\|\mathbf{J}_F^{(1)}\|_{L^2(F)}\|\nabla p\|_{L^2(\widetilde{T})}+\big\|\mathbf{J}_F^{(2)}\big\|_{L^2(F)}\|\nabla q\|_{L^2(\widetilde{T})}\right),\end{aligned}$$

which completes the proof. ☐

4.3. Proof of Theorem 3.1

As in [15], $R_1(x_3)$ and $R_2(x_3)$ are defined by

$$R_1(x_3)=\min\left\{\triangle_1^-\int_{b_1}^{x_3}s_2(t)dt,\triangle_1^+\int_{b_1}^{x_3}s_1(t)dt\right\},\quad x_3\ge b_1,$$

$$R_2(x_3)=\min\left\{\triangle_2^-\int_{x_3}^{b_2}s_2(t)dt,\triangle_2^+\int_{x_3}^{b_2}s_1(t)dt\right\},\quad x_3\le b_2.$$

The following two lemmas are concerned with the estimates for the extension, and can be respectively proven in the similar way as [15, Lemmas 4.3, 4.4] are done.

Lemma 4.9. *For any $v \in (X(\Omega))^2$, which is extended to be a vector function in $(X(D))^2$ as shown above, there holds*

$$\left\| s^{-1} e^{R_j(x_3)} \nabla v \right\|_{(L^2(\Omega_j^{PML}))^2} \le C_j \|v\|_{(H^1(\Omega))^2},$$

where

$$C_j = \widehat{C} \max\left(\frac{2k_j \delta_j^{\frac{1}{2}}}{1 - e^{-2\Delta_j^- \sigma_j^I}}, \frac{2\big(1 + 2\delta_j(\Delta_j^+ + k_j)\big)^{\frac{1}{2}}}{1 - e^{-2\Delta_j^+ \sigma_j^R}} \right), \quad j = 1, 2.$$

Lemma 4.10. *For any $v \in (X(\Omega))^2$, which is extended to be a vector function in $(X(D))^2$ as shown above, there holds*

$$\left\| s^{-1} \frac{\partial v}{\partial x_3} \right\|_{(L^2(\Gamma_1^{PML}))^2} \le \widehat{C} \mathbf{M}_3 \|v\|_{(H^1(\Omega))^2},$$

where

$$\mathbf{M}_3 = \max\left(\frac{2\,\Delta_1^-\, e^{-\Delta_1^- \sigma_1^I}}{1 - e^{-2\Delta_1^- \sigma_1^I}}, \frac{2\,\Delta_1^+\, e^{-\Delta_1^+ \sigma_1^R}}{1 - e^{-2\Delta_1^+ \sigma_1^R}} \right).$$

Now we are in the position to prove Theorem 3.1.

Proof. Denote by

$$\begin{aligned}
&\mathbb{J}^1 := \int_D g\overline{v}dx, && \mathbb{J}^2 := -A_D(\widetilde{u}_h, v),\\
&\mathbb{J}^3 := \frac{1}{\mu_1}\int_{\Gamma_1} \big(T^{(1)} - T^{(1,\mathrm{PML})}\big)(\widetilde{e}_h - e_{\mathrm{I}})\overline{p}dx_1, && \mathbb{J}^4 := \frac{1}{\mu_2}\int_{\Gamma_2} \big(T^{(2)} - T^{(2,\mathrm{PML})}\big)\widetilde{e}_h\overline{p}dx_1,\\
&\mathbb{J}^5 := \frac{1}{\varepsilon_1}\int_{\Gamma_1} \big(T^{(1)} - T^{(1,\mathrm{PML})}\big)(\widetilde{h}_h - h_{\mathrm{I}})\overline{q}dx_1, && \mathbb{J}^6 := \frac{1}{\varepsilon_2}\int_{\Gamma_2} \big(T^{(2)} - T^{(2,\mathrm{PML})}\big)\widetilde{h}_h\overline{q}dx_1,\\
&\mathbb{J}^7 := -\frac{1}{\mu_1}\int_{\Gamma_1^{\mathrm{PML}}} (e_{\mathrm{I}} - I_h e_{\mathrm{I}})\frac{1}{s(x_3)}\frac{\partial \overline{p}}{\partial x_3}dx_1, && \mathbb{J}^8 := -\frac{1}{\varepsilon_1}\int_{\Gamma_1^{\mathrm{PML}}} (h_{\mathrm{I}} - I_h h_{\mathrm{I}})\frac{1}{s(x_3)}\frac{\partial \overline{q}}{\partial x_3}dx_1.
\end{aligned}$$

It follows from the error representation formula (4.1) that

$$A(u - \widetilde{u}_h, v) := \sum_{i=1}^{8} \mathbb{J}^i.$$

Using integration by parts, (3.4)-(3.6) and Lemma 4.6, we obtain

$$\begin{aligned}
\mathbb{J}^1 + \mathbb{J}^2 = &\sum_{T\in\mathcal{M}_h} \Bigg(\int_T \mathbf{R}_T \cdot \overline{v}dx \\
&\qquad - \sum_{F\subset\partial T}\int_F \left(\frac{1}{\mu}\mathbb{A}\nabla\widetilde{e}_h \cdot n\overline{p} + \frac{1}{\varepsilon}\mathbb{A}\nabla\widetilde{h}_h \cdot n\overline{q} - i\omega\beta\nabla\widetilde{h}_h \cdot n\overline{p} + i\omega\beta\nabla\widetilde{e}_h \cdot n\overline{q} \right) ds \Bigg)\\
= &\sum_{T\in\mathcal{M}_h} \left(\int_T \mathbf{R}_T \cdot (\overline{v} - \mathcal{I}_h^*\overline{v}_h)dx - \sum_{F\subset\partial T}\frac{1}{2}\int_F \left(\mathbf{J}_F^{(1)}(\overline{p} - \mathcal{I}_h^*\overline{p}_h) + \mathbf{J}_F^{(2)}(\overline{q} - \mathcal{I}_h^*\overline{q}_h) \right) ds \right).
\end{aligned}$$

From Lemmas 4.7, 4.8, (3.6) and Lemma 4.9, we have

$$
\begin{aligned}
|\mathbb{J}^1+\mathbb{J}^2| &\le C\sum_{T\in\mathcal{M}_h}\eta_T\|w^{-1}\nabla v\|_{(L^2(\widetilde{T}))^2}\\
&\le C(1+C_1+C_2)\left(\sum_{T\in\mathcal{M}_h}\eta_T^2\right)^{\frac{1}{2}}\|v\|_{(H^1(\Omega))^2}. \qquad (4.7)
\end{aligned}
$$

Next, we estimate the term $\mathbb{J}^3$. Employing Lemma 4.4 yields

$$
\mathbb{J}^3\le \mathbf{M}_1\|\widetilde{e}_h-e_I\|_{L^2(\Gamma_1)}\|p\|_{L^2(\Gamma_1)},
$$

similarly, there holds

$$
\begin{aligned}
&\mathbb{J}^4\le \mathbf{M}_2\|\widetilde{e}_h\|_{L^2(\Gamma_2)}\|p\|_{L^2(\Gamma_2)},\\
&\mathbb{J}^5\le \mathbf{M}_1\big\|\widetilde{h}_h-h_{\mathrm{I}}\big\|_{L^2(\Gamma_1)}\|q\|_{L^2(\Gamma_1)},\\
&\mathbb{J}^6\le \mathbf{M}_2\big\|\widetilde{h}_h\big\|_{L^2(\Gamma_2)}\|q\|_{L^2(\Gamma_2)}.
\end{aligned}
$$

Combining the above four inequality with Lemma 4.1 gives

$$
\left|\sum_{i=3}^{6}\mathbb{J}^i\right|\le \widehat{C}\left(\mathbf{M}_1\|\widetilde{u}_h-u_I\|_{(L^2(\Gamma_1))^2}+\mathbf{M}_2\|\widetilde{u}_h\|_{(L^2(\Gamma_2))^2}\right)\|v\|_{(H^1(\Omega))^2}. \qquad (4.8)
$$

It remains to estimate the term $\mathbb{J}^7$ and $\mathbb{J}^8$. Applying Lemmas 4.1 and 4.10, we have

$$
\begin{aligned}
|\mathbb{J}^7+\mathbb{J}^8| &\le C\Bigg(\|e_I-I_he_{\mathrm{I}}\|_{L^2(\Gamma_1^{\mathrm{PML}})}\left\|s^{-1}\frac{\partial p}{\partial x_3}\right\|_{L^2(\Gamma_1^{\mathrm{PML}})}\\
&\qquad +\|h_I-I_hh_{\mathrm{I}}\|_{L^2(\Gamma_1^{\mathrm{PML}})}\left\|s^{-1}\frac{\partial q}{\partial x_3}\right\|_{L^2(\Gamma_1^{\mathrm{PML}})}\Bigg)\\
&\le \widehat{C}\mathbf{M}_3\|u_{\mathrm{I}}-I_hu_{\mathrm{I}}\|_{(L^2(\Gamma_1^{\mathrm{PML}}))^2}\|v\|_{(H^1(\Omega))^2}. \qquad (4.9)
\end{aligned}
$$

Finally, the proof of Theorem 3.1 follows by combining (4.7)-(4.9) and the inf-sup condition (2.11). □

4.4. Local lower bound

In this subsection, we establish a posteriori lower bound for $\widehat{u}-\widetilde{u}_h$ in the H^1 norm. To obtain the local lower bound, we need to use the arguments similar to those in [19] with the aid of the bubble functions.

To proceed, we introduce some notations. Let $\mathbf{R}_T^a$ be the integral mean of $\mathbf{R}_T$ on T, and $\mathbf{J}_F^a$ be the integral mean of $\mathbf{J}_F$ on $F, i=1,2$.

Theorem 4.1. *There exist constants C_3, C_4 and C_5, depending on the minimum angle of $\mathcal{M}_h$ and the maximum value of $w(x_3)$ such that for any $T\in\mathcal{M}_h$, there holds*

$$
\begin{aligned}
\eta_T^2 &\le C_3\|\widehat{u}-\widetilde{u}_h\|^2_{(H^1(T^*))^2}+C_4\sum_{T\subset T^*}h_T^2\big\|\mathbf{R}_T-\mathbf{R}_T^a\big\|^2_{(L^2(T))^2}\\
&\quad +C_5\sum_{F\subset\partial T}h_F\big\|\mathbf{J}_F-\mathbf{J}_F^a\big\|^2_{(L^2(F))^2},
\end{aligned}
$$

where T^* consists of all elements sharing at least one common side with T.

Proof. In the following, the proof is divided into three steps.

Step 1: Interior residual. Let $b_T = 27\lambda_1\lambda_2\lambda_3$ be the bubble function, which is supported in T and satisfies

$$\big\|b_T\mathbf{R}_T^a\big\|_{(L^2(T))^2} + h_T\big\|\nabla\big(b_T\mathbf{R}_T^a\big)\big\|_{(L^2(T))^2} \le C\big\|\mathbf{R}_T^a\big\|_{(L^2(T))^2}. \tag{4.10}$$

Using (2.15), integration by parts, the Cauchy-Schwarz inequality and (4.10), we have

$$\begin{aligned}
C\big\|\mathbf{R}_T^a\big\|^2_{(L^2(T))^2} &\le \big(\mathbf{R}_T^a, b_T\mathbf{R}_T^a\big) = \big(\mathbf{R}_T, b_T\mathbf{R}_T^a\big) - \big(\mathbf{R}_T - \mathbf{R}_T^a, b_T\mathbf{R}_T^a\big)\\
&= -A_T\big(\widetilde{u}_h, b_T\mathbf{R}_T^a\big) + \big(g, b_T\mathbf{R}_T^a\big) - \big(\mathbf{R}_T - \mathbf{R}_T^a, b_T\mathbf{R}_T^a\big)\\
&= A_T\big(\widehat{u} - \widetilde{u}_h, b_T\mathbf{R}_T^a\big) - \big(\mathbf{R}_T - \mathbf{R}_T^a, b_T\mathbf{R}_T^a\big)\\
&\le C\Big(\|\nabla(\widehat{u}-\widetilde{u}_h)\|_{(L^2(T))^2}\big\|\nabla\big(b_T\mathbf{R}_T^a\big)\big\|_{(L^2(T))^2} + \|\widehat{u}-\widetilde{u}_h\|_{(L^2(T))^2}\big\|b_T\mathbf{R}_T^a\big\|_{(L^2(T))^2}\\
&\qquad + \big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|_{(L^2(T))^2}\big\|b_T\mathbf{R}_T^a\big\|_{(L^2(T))^2}\Big)\\
&\le C\left(h_T^{-1}\|\widehat{u} - \widetilde{u}_h\|_{(H^1(T))^2} + \big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|_{(L^2(T))^2}\right)\big\|\mathbf{R}_T^a\big\|_{(L^2(T))^2},
\end{aligned}$$

which implies that

$$\big\|\mathbf{R}_T^a\big\|_{(L^2(T))^2} \le C\left(h_T^{-1}\|\widehat{u} - \widetilde{u}_h\|_{(H^1(T))^2} + \big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|_{(L^2(T))^2}\right).$$

It follows from the triangle inequality and the above estimate that

$$\begin{aligned}
h_T^2\|\mathbf{R}_T\|^2_{(L^2(T))^2} &\le h_T^2\big\|\mathbf{R}_T^a\big\|^2_{(L^2(T))^2} + h_T^2\big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|^2_{(L^2(T))^2}\\
&\le C\left(\|\widehat{u} - \widetilde{u}_h\|^2_{(H^1(T))^2} + h_T^2\big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|^2_{(L^2(T))^2}\right).
\end{aligned} \tag{4.11}$$

Step 2: Jump residual. For any interior side $F = \partial T_1 \bigcap \partial T_2$, let $b_F = 4\lambda_1\lambda_2$ be an edge bubble function, which is supported in $\mathbf{w}_F = T_1 \cup T_2$ and satisfies

$$h_F^{-\frac{1}{2}}\big\|b_F\mathbf{J}_F^a\big\|_{(L^2(T_i))^2} + h_F^{\frac{1}{2}}\big\|\nabla\big(b_F\mathbf{J}_F^a\big)\big\|_{(L^2(T_i))^2} \le C\big\|\mathbf{J}_F^a\big\|_{(L^2(F))^2}, \quad i = 1, 2. \tag{4.12}$$

From (2.15), integration by parts, the Cauchy-Schwarz inequality and (4.12), we obtain

$$\begin{aligned}
C\big\|\mathbf{J}_F^a\big\|^2_{(L^2(F))^2} &\le \big(\mathbf{J}_F^a, b_F\mathbf{J}_F^a\big) = \big(\mathbf{J}_F, b_F\mathbf{J}_F^a\big) - \big(\mathbf{J}_F - \mathbf{J}_F^a, b_F\mathbf{J}_F^a\big)\\
&= -\int_{\mathbf{w}_F}\mathbf{R}_T\cdot b_F\mathbf{J}_F^a dx - A_{\mathbf{w}_F}\big(\widetilde{u}_h, b_F\mathbf{J}_F^a\big) + \big(g, b_F\mathbf{J}_F^a\big) - \big(\mathbf{J}_F - \mathbf{J}_F^a, b_F\mathbf{J}_F^a\big)\\
&= -\int_{\mathbf{w}_F}\mathbf{R}_T\cdot b_F\mathbf{J}_F^a dx + A_{\mathbf{w}_F}\big(\widehat{u}-\widetilde{u}_h, b_F\mathbf{J}_F^a\big) - \big(\mathbf{J}_F-\mathbf{J}_F^a, b_F\mathbf{J}_F^a\big)\\
&\le C\Big(\|\mathbf{R}_T\|_{(L^2(\mathbf{w}_F))^2}\big\|b_F\mathbf{J}_F^a\big\|_{(L^2(\mathbf{w}_F))^2} + \|\nabla(\widehat{u}-\widetilde{u}_h)\|_{(L^2(\mathbf{w}_F))^2}\big\|\nabla(b_F\mathbf{J}_F^a)\big\|_{(L^2(\mathbf{w}_F))^2}\\
&\qquad + \|\widehat{u}-\widetilde{u}_h\|_{(L^2(\mathbf{w}_F))^2}\big\|b_F\mathbf{J}_F^a\big\|_{(L^2(\mathbf{w}_e))^2} + \big\|\mathbf{J}_F-\mathbf{J}_F^a\big\|_{(L^2(F))^2}\big\|b_F\mathbf{J}_F^a\big\|_{(L^2(F))^2}\Big)\\
&\le C\Big(h_F^{\frac{1}{2}}\|\mathbf{R}_T\|_{(L^2(\mathbf{w}_F))^2} + h_F^{-\frac{1}{2}}\|\widehat{u} - \widetilde{u}_h\|_{(H^1(\mathbf{w}_F))^2}\\
&\qquad + h_F^{\frac{1}{2}}\big\|\mathbf{J}_F - \mathbf{J}_F^a\big\|_{(L^2(F))^2}\Big)\|\mathbf{J}_F^a\|_{(L^2(F))^2},
\end{aligned}$$

which implies that

$$\big\|\mathbf{J}_F^a\big\|_{(L^2(F))^2} \le C\Big(h_F^{\frac{1}{2}}\|\mathbf{R}_T\|_{(L^2(\mathbf{w}_F))^2} + h_F^{-\frac{1}{2}}\|\widehat{u} - \widetilde{u}_h\|_{(H^1(\mathbf{w}_F))^2} + h_F^{\frac{1}{2}}\big\|\mathbf{J}_F - \mathbf{J}_F^a\big\|_{(L^2(F))^2}\Big).$$

Combining the above estimate and the triangle inequality, we obtain

$$\begin{aligned}h_F^{\frac{1}{2}}\|\mathbf{J}_F\|_{(L^2(F))^2} &\le h_F^{\frac{1}{2}}\big\|\mathbf{J}_F^a\big\|_{(L^2(F))^2} + h_F^{\frac{1}{2}}\big\|\mathbf{J}_F - \mathbf{J}_F^a\big\|_{(L^2(F))^2} \\ &\le C\Big(h_T\|\mathbf{R}_T\|_{L^2(\mathbf{w}_F)} + \|\widehat{u} - \widetilde{u}_h\|_{(H^1(\mathbf{w}_F))^2} + h_F^{\frac{1}{2}}\big\|\mathbf{J}_F - \mathbf{J}_F^a\big\|_{(L^2(F))^2}\Big). \qquad (4.13)\end{aligned}$$

For any side $F = \Gamma_{left} \cap \partial T_1$, there is a corresponding side F' on Γ_{right} which belongs to some element T_2, and vice versa. We can use the similar argument as above to prove (4.13) for $F \subset \Gamma_{left}$ or Γ_{right}.

Step 3: Local lower bound. Therefore, by (4.11) and (4.13), we have

$$\begin{aligned}\eta_T^2 &= \left(\max_{x\in\widetilde{T}} \mathbf{w}(x_3)\left\{h_T\|\mathbf{R}_T\|_{(L^2(T))^2} + \left(\frac{1}{2}\sum_{F\subset\partial T} h_F\|\mathbf{J}_F\|^2_{(L^2(F))^2}\right)^{\frac{1}{2}}\right\}\right)^2 \\ &\le C_3\|\widehat{u} - \widetilde{u}_h\|^2_{(H^1(T^*))^2} + C_4\sum_{T\subset T^*} h_T^2\big\|\mathbf{R}_T - \mathbf{R}_T^a\big\|^2_{(L^2(T))^2} \\ &\quad + C_5\sum_{F\subset\partial T} h_F\big\|\mathbf{J}_F - \mathbf{J}_F^a\big\|^2_{(L^2(F))^2}.\end{aligned}$$

This completes the proof. □

5. Numerical Experiments

In this section, two numerical examples(cf. [36]) are presented to validate our theoretical findings and the effectiveness of our adaptive PML finite volume method. The implementation of the adaptive finite volume algorithm is based on the PDE toolbox of MATLAB. We note that the a posteriori error estimate from Theorem 3.1 is used to determine the PML parameters. Using similar implementation as [15, Section 6], we choose the PML medium property $s(x_3)$ as the power function, and choose the thickness $\delta = \delta_1 = \delta_2$ of the PML layers and the medium parameters σ_j satisfying $\mathbf{M}_j L^{1/2} \le 10^{-8}$ such that the PML error is negligible compared with the finite volume discretization error, $j = 1, 2$. Once the PML region and the medium are fixed, the adaptive finite volume strategy is designed to modify the mesh. Our adaptive PML finite volume algorithm, which is similar to the adaptive PML finite element algorithm in [15], is omitted in this paper.

Example 5.1. We consider a chiral grating with two sharp angles (see Fig. 5.1). Assume that the plane waves

$$e_I = 0.8e^{i\alpha x_1 - i\beta_1 x_3}, \quad h_I = 0.6e^{i\alpha x_1 - i\beta_1 x_3}$$

is incidence on the structure with $L=1, b_1=0.5$ and $b_2=-0.5$, where $\theta=\pi/6$ and $\omega = 2.5$. The other parameters are chosen as $\varepsilon_1 = \varepsilon_4 = 1$, $\varepsilon_2 = 2.56$, $\varepsilon_3 = 4.84$, $\beta_2 = 0.2$ and $\beta_3 = 0.1$. The thickness of the PML is set to be $\delta = 1$. In Fig. 5.2(a), the grating efficiency of the reflected and transmitted waves as well as the total grating efficiency are displayed as a function of the number of nodal points (DoFs) of adaptive refined meshes. It is clear that the efficiencies are convergence for our adaptive PML finite volume algorithm. The adaptively refined mesh and the amplitude of the numerical solutions of the electric field and magnetic field are illustrated in Figs. 5.3(a) and 5.4 when the mesh has 11358 DoFs. It can be seen that although there is a difference in

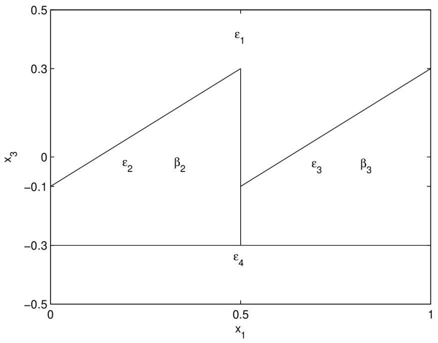

Fig. 5.1. Geometry of the domain for Example 5.1.

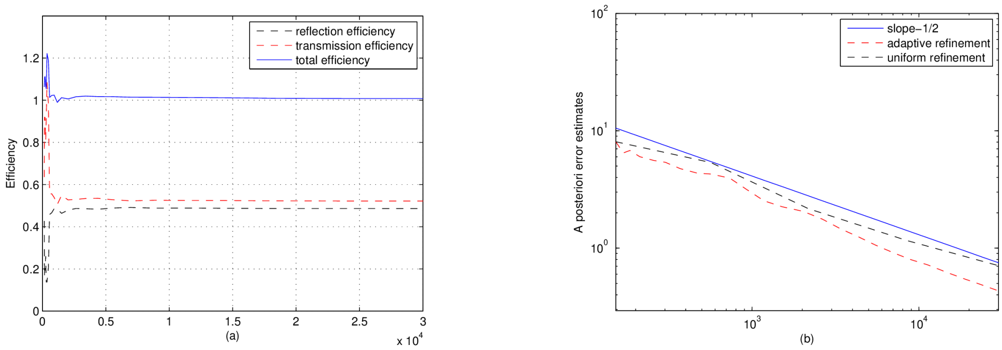

Fig. 5.2. Grating efficiency versus DoFs (a) and log-log plot of the a posteriori error estimates with respect to DoFs (b) for Example 5.1.

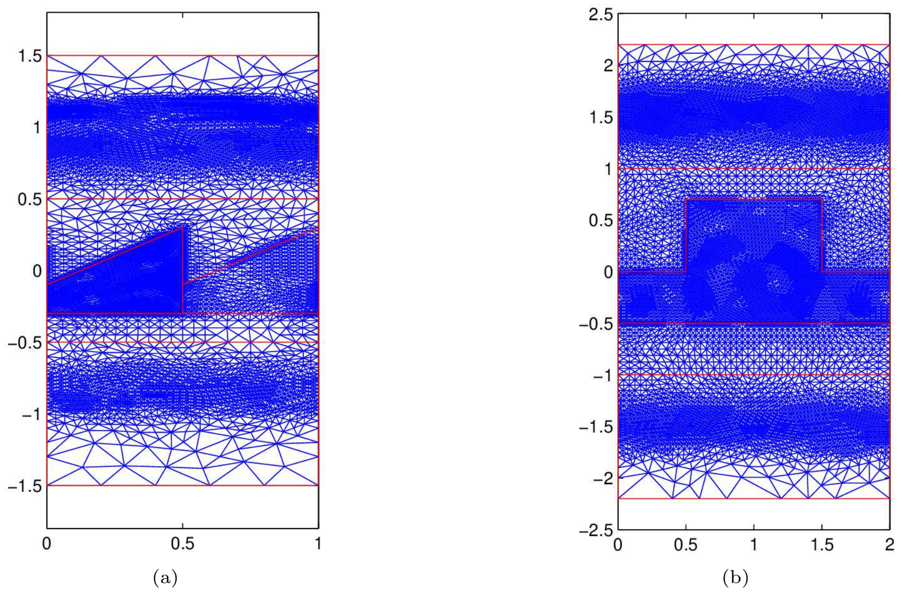

Fig. 5.3. (a): An adaptively refined mesh with 11358 DoFs for Example 5.1, (b): An adaptively refined mesh with 12256 DoFs for Example 5.2.

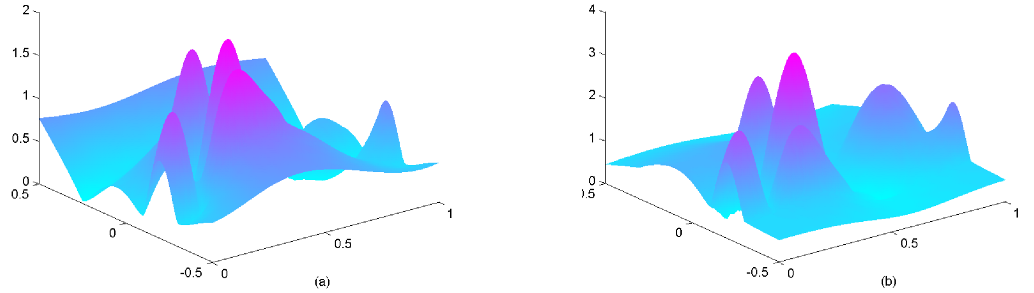

Fig. 5.4. The surface plot of the amplitude of the electric field (a) and the magnetic field (b) on the mesh in Fig. 5.3(a) for Example 5.1.

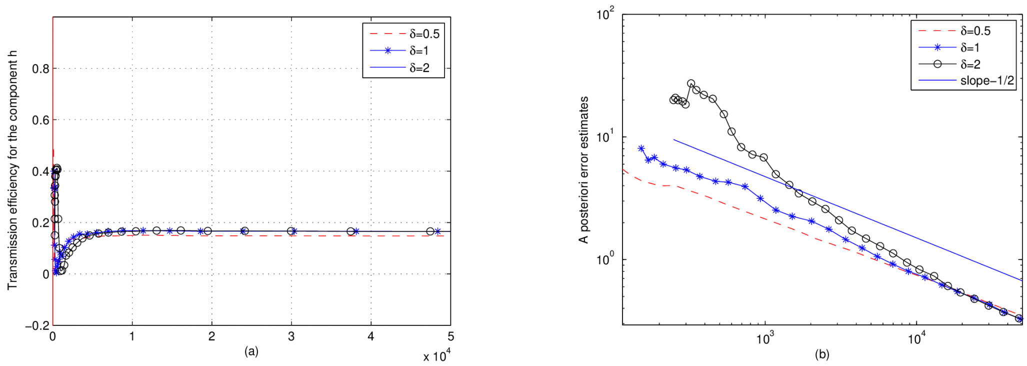

Fig. 5.5. Robustness of the transmission efficiency for the component h (a) and quasi-optimality of the posteriori error estimates (b) with respect to the thickness of PML layers for Example 5.1.

the meshes, the surface plots of the amplitude of the associated solutions for our adaptive PML finite volume method is completely similar to that of the PML finite element method(cf. [36]). Fig. 5.2(b) shows the log-log plot of the a posteriori error estimates $\epsilon_h = (\sum_{T\in\mathcal{M}_h} \eta_T^2)^{1/2}$ with respect to DoFs which indicates that the mesh and the associated numerical complexity for our adaptive method are quasi-optimal: $\epsilon_h = \mathcal{O}(\text{DoFs}^{-1/2})$ is valid asymptotically. Here we mention that our adaptive method meets the the principle that the finite volume discretization error ϵ_h is not contaminated by the truncation error of the exponentially decaying factor $\mathbf{M}_j, j = 1, 2, 3,$ in Theorem 3.1. Fig. 5.5 shows the curves of the transmission efficiency for the component h versus DoFs and the curves of $\log\epsilon_h$ versus logDoFs for the thickness $\delta = 0.5, 1, 2$ of PML layers. It clearly demonstrates that our adaptive method is robust with respect to the choice of the thickness of PML layers: the transmission efficiency for h are convergence and insensitive to the thickness δ, and the meshes and the associated numerical complexity are quasi-optimal for the different choice of δ.

Example 5.2. This example concerns a chiral grating whose surface has corners, as seen in Fig. 5.6. The parameters are taken as follows: $\varepsilon_1 = 1, \varepsilon_2 = 2.25, \varepsilon_3 = 1, \beta_2 = 0.1$, and $L = 2$. The incident plane waves are $e_I = e^{i\alpha x_1 - i\beta_1 x_3}, h_I = 0$ with $\omega = \pi$ and $\theta = \pi/4$. We set $\delta = 1.2$. Fig. 5.7(a) shows the reflection efficiency, the transmission efficiency, and the total grating efficiency as a function of DoFs. The mesh with 12256 DoFs and the amplitude of the electric field and magnetic field are presented in Figs. 5.3(b) and 5.8, respectively. Just like

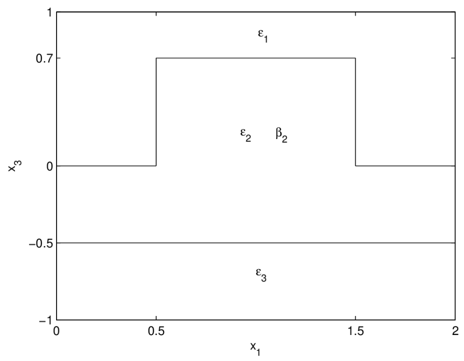

Fig. 5.6. Geometry of the domain for Example 5.2.

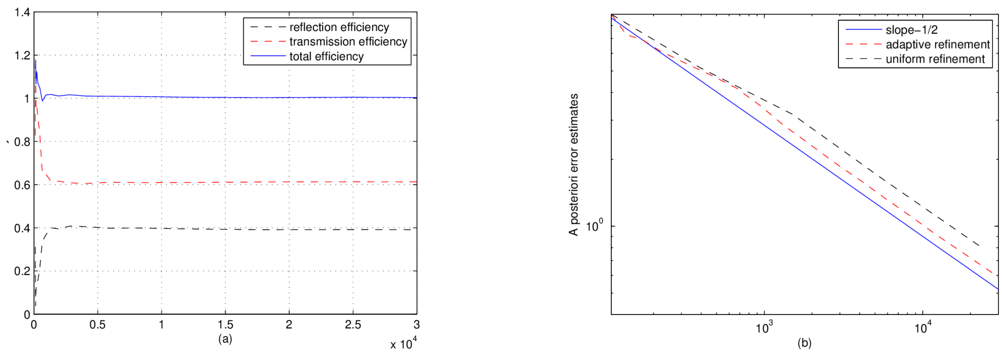

Fig. 5.7. Grating efficiency versus DoFs (a) and log-log plot of the a posteriori error estimates with respect to DoFs (b) for Example 5.2.

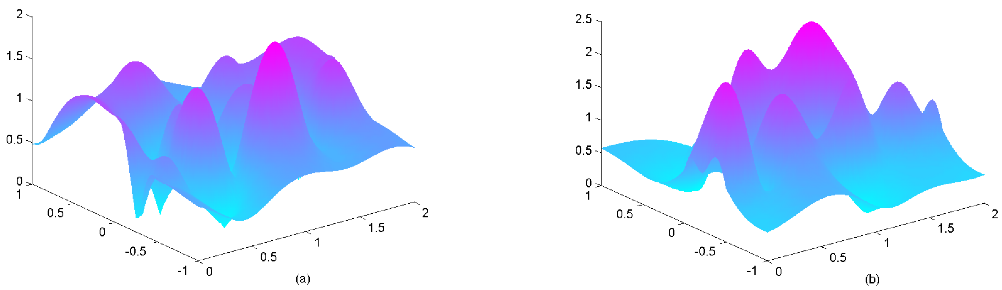

Fig. 5.8. The surface plot of the amplitude of the electric field (a) and the magnetic field (b) on the mesh in Fig. 5.3(b) for Example 5.2.

Example 5.1, it is observed that the a posteriori error has the ability to catch the singularities of the solution to the problem (2.7) by using the local grid refinement. Fig. 5.7(b) displays the log-log plot between the a posteriori error estimates ϵ_h and DoFs, and it can be seen that $\epsilon_h = \mathcal{O}(\text{DoFs}^{-1/2})$ is valid asymptotically. Fig. 5.9 shows the curves of the reflection efficiency for the component e versus DoFs and the logϵ_h- logDoFs curves for the thickness $\delta = 0.6, 1.2, 2$ of PML layers, which further indicates that our adaptive method works effectively and is robust with respect to the choice of the thickness δ.

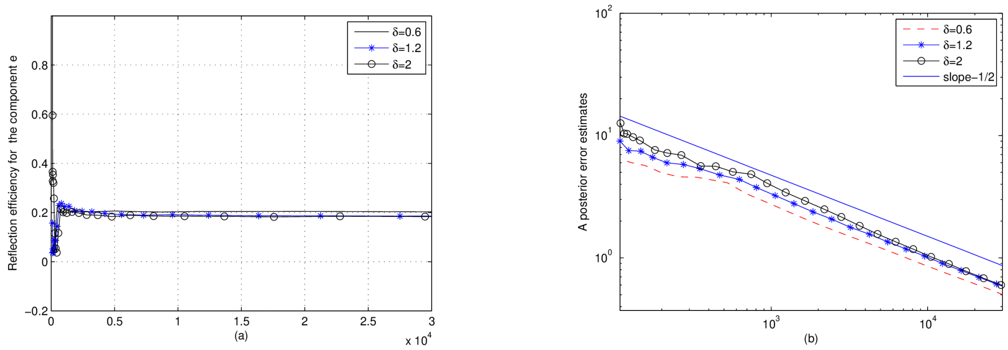

Fig. 5.9. Robustness of the reflection efficiency for the component e (a) and quasi-optimality of the posteriori error estimates (b) with respect to the thickness of PML layers for Example 5.2.

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