POSITIVE DEFINITE AND SEMI-DEFINITE SPLITTING METHODS FOR NON-HERMITIAN POSITIVE DEFINITE LINEAR SYSTEMS

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Abstract

In this paper, we further generalize the technique for constructing the normal (or positive definite) and skew-Hermitian splitting iteration method for solving large sparse non-Hermitian positive definite system of linear equations. By introducing a new splitting, we establish a class of efficient iteration methods, called positive definite and semi-definite splitting (PPS) methods, and prove that the sequence produced by the PPS method converges unconditionally to the unique solution of the system. Moreover, we propose two kinds of typical practical choices of the PPS method and study the upper bound of the spectral radius of the iteration matrix. In addition, we show the optimal parameters such that the spectral radius achieves the minimum under certain conditions. Finally, some numerical examples are given to demonstrate the effectiveness of the considered methods.

Key words: Linear systems, Splitting method, Non-Hermitian matrix, Positive definite matrix, Positive semi-definite matrix, Convergence analysis.

1. Introduction

Many problems in scientific computing give rise to a system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{C}^{n\times n}, \quad \text{and} \quad \mathbf{x}, \mathbf{b} \in \mathbb{C}^n,$$

with $\mathbf{A}$ being a large sparse non-Hermitian but positive definite matrix.

We call a matrix $\mathbf{B}$ positive definite (or positive semi-definite), if $\mathbf{B} + \mathbf{B}^*$ is Hermitian positive definite (or positive semi-definite), i.e., for all $0 \neq \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* (\mathbf{B} + \mathbf{B}^*) \mathbf{x} > 0$ (or $\mathbf{x}^* (\mathbf{B} + \mathbf{B}^*) \mathbf{x} \geq 0$), where $\mathbf{B}^*$ denotes the complex conjugate transpose of the $\mathbf{B}$. Let $\mathbf{D} = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ be the diagonal part of $\mathbf{A}$ and $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$. Since the coefficient matrix $\mathbf{A}$ is positive definite, we have $\mathbf{e}_i^* (\mathbf{A} + \mathbf{A}^*) \mathbf{e}_i = a_{ii} + \sigma_i^2 > 0$. This shows that $\mathbf{D}$ is positive definite.

The linear system has many important practical applications, such as diffuse optical tomography, molecular scattering, lattice quantum chromodynamics (see, e.g., [1,7,8,22,24,38,39,41]). Many researchers have been devoted themselves to the numerical solution of (1.1) (see e.g., [2–4,10,11,18,21,25,27,28,36,37,40,42,45–47] and the references therein) and proposed kinds of available iteration methods for solving the system (1.1), in which splitting iteration methods (see e.g., [9,13–17,19,29–31,35,44]) and Krylov subspace methods (see e.g., [5,20,23,26,32,43]).

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attract a lot of attention. In [16], the authors presented a Hermitian and skew-Hermitian splitting (HSS) iteration method for solving (1.1) and showed that the HSS method converges unconditionally to the unique solution of the system. Then many researchers focused on the HSS method and proposed kinds of iterations method based on Hermitian and skew-Hermitian splitting (see e.g., [6, 9, 12, 13, 17]). In recent years, other kinds of splitting iteration methods have also been studied (see e.g., [14,15,33–35,44]). Normal and skew-Hermitian splitting (NSS) iteration methods for solving large sparse non-Hermitian positive definite linear system was studied in [15]. Based on block triangular and skew-Hermitian splitting, a class of iteration methods for solving positive-definite linear systems was established in [14]. Krukier et. al proposed the generalized skew-Hermitian triangular splitting iteration methods to solve (1.1) and applied the methods to solve the saddle-point linear systems (see [34]). In this work, we further generalize the technique for constructing the normal (or positive definite) and skew-Hermitian splitting iteration method for solving (1.1).

Throughout this paper, we use the following notations: \( \mathbb{C}^{m \times n} \) is the set of \( m \times n \) complex matrices and \( \mathbb{C}^{m} = \mathbb{C}^{m \times 1} \). We use \( C \) and \( R \) to denote the set of complex numbers and real numbers, respectively. For any \( a \in C \), we write \( \text{Re}(a) \) and \( \text{Im}(a) \) to denote the real and imaginary parts of \( a \). For \( B \in \mathbb{C}^{n \times n} \), we write \( B^{-1} \), \( \|B\|_2 \), \( \Lambda(B) \) and \( \rho(B) \) to denote the the inverse, 2-norm, the spectrum and the spectral radius of the matrix \( B \), respectively. \( I \) denotes the identity matrix of size implied by context. \( i = \sqrt{-1} \) denotes the imaginary unit.

The organization of this paper is as follows. In Section 2, we present the positive definite and semi-definite splitting methods for solving non-hermitian positive definite linear systems and study the convergence properties of the PPS iteration. In Section 3, we establish two kinds of typical practical choices of the PPS method and study the upper bound of the spectral radius of iteration matrix. Numerical experiments are presented in Section 4 to show the effectiveness of our methods.

2. The Positive Definite and Semi-definite Splitting Method

In this section, we study efficient iterative methods for solving (1.1) based on the positive definite and semi-definite splitting (PPS for short) of the coefficient matrix \( A \), and establish the convergence analysis of the new methods. For this purpose, we split \( A \) into positive-definite and positive semi-definite parts as follows:

\[
A = M + N,
\]

where \( M \) is a positive-definite matrix and \( N \) is a positive semi-definite matrix. Then it is easy to see that

\[
A = \alpha I + M - (\alpha I - N) = \alpha I + N - (\alpha I - M).
\]

This implies that the system (1.1) can be reformulated equivalently as:

\[
(\alpha I + M)x = (\alpha I - N)x + b,
\]

or

\[
(\alpha I + N)x = (\alpha I - M)x + b.
\]

By the two above fixed point equations, we can get the following iteration method:

\[
\begin{cases}
(\alpha I + M)x^{(k+\frac{1}{2})} = (\alpha I - N)x^{(k)} + b, \\
(\alpha I + N)x^{(k+1)} = (\alpha I - M)x^{(k+\frac{1}{2})} + b.
\end{cases}
\]

\hspace{1cm} (2.2)
For convenience, we call the scheme (2.2) positive-definite and semi-definite splitting (PPS for short) iteration method. It is worth mentioning that the normal and skew-Hermitian splitting iteration method proposed in [15] and the positive definite and skew-Hermitian splitting iteration method proposed in [14] are both the special cases of the PPS iteration method. Actually, if \( A = M + N \) with \( N \) being a skew-Hermitian matrix, then we can see that

\[
M + M^* = A - N + (A - N)^* = A + A^*,
\]

which shows that \( M \) is positive definite. This, along with the fact that all the skew-Hermitian matrix are positive semi-definite, follows that both of the two splitting mentioned above belong to the positive-definite and semi-definite splitting.

In the following, we shall establish the convergence theorem of the PPS method. To this end, we rewrite the scheme (2.2) equivalently. By using (2.2), we can obtain

\[
x^{(k+\frac{1}{2})} = (\alpha I + M)^{-1}(\alpha I - N)x^{(k)} + (\alpha I + M)^{-1}b, \tag{2.3}
\]

and

\[
x^{(k+1)} = (\alpha I + N)^{-1}(\alpha I - M)x^{(k+\frac{1}{2})} + (\alpha I + N)^{-1}b, \tag{2.4}
\]

which follows that

\[
x^{(k+1)} = S(\alpha)x^{(k)} + T(\alpha)b, \tag{2.5}
\]

where

\[
S(\alpha) = (\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N), \tag{2.6a}
\]

\[
T(\alpha) = 2\alpha[(\alpha I + M)(\alpha I + N)]^{-1}. \tag{2.6b}
\]

Therefore, the PPS method is convergent if and only if the spectral radius of the iterative matrix \( S(\alpha) \) is less than 1. To this purpose, we present the following lemma.

**Lemma 2.1.** For any positive semi-definite matrix \( P \in \mathbb{C}^{n \times n} \), it holds that

\[
\|((\alpha I - P)(\alpha I + P)^{-1}\|_2 \leq 1, \quad \forall \alpha > 0.
\]

Furthermore, if \( P \) is positive definite, then it holds that

\[
\|((\alpha I - P)(\alpha I + P)^{-1}\|_2 < 1, \quad \forall \alpha > 0.
\]

**Proof.** Let \( \sigma(\alpha) = \|((\alpha I - P)(\alpha I + P)^{-1}\|_2 \). By the definition of 2-norm and the similarity invariance of the matrix spectrum, we have

\[
\sigma(\alpha)^2 = \|((\alpha I - P)(\alpha I + P)^{-1}\|_2^2 = \rho((\alpha I + P^*)^{-1}(\alpha I - P^*)(\alpha I - P)(\alpha I + P)^{-1}) = \rho((\alpha I - P^*)(\alpha I - P)(\alpha I + P)^{-1})(\alpha I + P^*)^{-1}. \]

Note that the matrix \((\alpha I + P^*)^{-1}(\alpha I - P^*)(\alpha I - P)(\alpha I + P)^{-1}\) is Hermitian positive semi-definite. It follows that all the eigenvalues of \((\alpha I - P^*)(\alpha I - P)(\alpha I + P)^{-1}(\alpha I + P^*)^{-1}\) are
nonnegative. This, together with the fact

\[
(\alpha I - P^*)(\alpha I - P)(\alpha I + P)^{-1}(\alpha I + P^*)^{-1} \\
= [\alpha^2 I + P^*P - \alpha(P + P^*)][\alpha^2 I + P^*P + \alpha(P + P^*)]^{-1} \\
= [\alpha^2 I + P^*P + \alpha(P + P^*) - 2\alpha(P + P^*)][\alpha^2 I + P^*P + \alpha(P + P^*)]^{-1} \\
= I - 2\alpha(P + P^*)[\alpha^2 I + P^*P + \alpha(P + P^*)]^{-1} \\
=: I - L(\alpha),
\]

yields that all the eigenvalues of the matrix \( I - L(\alpha) \) are nonnegative and

\[
\sigma(\alpha)^2 = \rho(I - L(\alpha)). \tag{2.7}
\]

For any \( \lambda \in \Lambda(L(\alpha)) \), it is easy to see that \( 1 - \lambda \in \Lambda(I - L(\alpha)) \), which shows that \( 1 - \lambda \geq 0 \). Then for any positive semi-definite matrix \( P \) and \( \alpha > 0 \), we can know that \( 2\alpha(P + P^*) \) is Hermitian positive semi-definite. This, along with the fact that \( L(\alpha) \) is similar to \( 2\alpha(\alpha I + P^*)^{-1}(P + P^*)(\alpha I + P)^{-1} \), leads to \( \lambda \geq 0 \). Then we have

\[
0 \leq 1 - \lambda \leq 1.
\]

Combining this with (2.7), it leads to

\[
\sigma(\alpha)^2 = \max_{\lambda \in \Lambda(L(\alpha))} 1 - \lambda \leq 1.
\]

Furthermore, if \( P \) is positive definite, then it is easy to see that \( 2\alpha(\alpha I + P^*)^{-1}(P + P^*)(\alpha I + P)^{-1} \) is Hermitian positive definite, which shows that \( \lambda > 0 \). At this case, we have \( 0 \leq 1 - \lambda < 1 \). It leads to \( \sigma(\alpha)^2 < 1 \) immediately. Then the result follows.

Now we establish the convergence theorem for the PPS iteration method.

**Theorem 2.1.** Let \( A = M + N \) be a positive definite and semi-definite splitting of \( A \). Then for any positive constant \( \alpha \), the spectral radius \( \rho(S(\alpha)) \) of the iteration matrix \( S(\alpha) \) satisfies \( \rho(S(\alpha)) < 1 \), where \( S(\alpha) \) is defined as in (2.6a). Therefore, the PPS iteration method is convergent to the exact solution \( x^* \in \mathbb{C}^n \) of the linear system (1.1).

**Proof.** Note that \( N \) is a positive semi-definite matrix and \( M \) is a positive definite matrix. Then by using Lemma 2.1, we can get

\[
\rho(S(\alpha)) = \rho((\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)) \\
= \rho((\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)(\alpha I + N)^{-1}) \\
\leq \|[\alpha I - M](\alpha I + M)^{-1}\|_2\|[\alpha I - N](\alpha I + N)^{-1}\|_2 \\
\leq \|[\alpha I - M](\alpha I + M)^{-1}\|_2 < 1,
\]

which completes the proof.

We shall emphasize that the scheme (2.5) can induce a class of preconditioners for solving the system (1.1). Actually, let

\[
R(\alpha) = \frac{1}{2\alpha}(\alpha I - M)(\alpha I - N),
\]
where \( M, N \) are defined as in (2.1). Then by using the definition of \( T(\alpha) \), we can know that
\[
A = T(\alpha)^{-1} - R(\alpha)
\] (2.8)
defines a new splitting of the coefficient matrix \( A \) in (1.1). It is easy to see that the PPS iteration method can also be induced by the matrix splitting (2.8). Moreover, the matrix \( T(\alpha)^{-1} \) can be alternatively used to precondition the Krylov subspace iteration methods for solving the system (1.1).

3. Two Typical Practical Choices of the PP-splitting

In this section, we give two kinds of typical practical choices of the PPS method. One is based on Hermitian and skew-Hermitian splitting, the other one is based on triangular splitting. To this end, let
\[
H(A) = \frac{1}{2}(A + A^*), \quad S(A) = \frac{1}{2}(A - A^*).
\]
Then \( H(A), S(A) \) are the Hermitian and skew-Hermitian parts of the matrix \( A \), respectively.

Let \( \eta \) be any real constant, let
\[
M = H(A) + i\eta I, \quad N = S(A) - i\eta I.
\] (3.1)
It is easy to verify that \( M + N = A, M + M^* = H(A) + i\eta I + H(A) - i\eta I = 2H(A) \) is positive definite, and \( N + N^* = S(A) - i\eta I - S(A) + i\eta I = 0 \) is positive semi-definite. Thereby, we can get the following PPS iteration method:
\[
\begin{aligned}
\left\{
(\alpha I + H(A) + i\eta I)x^{(k+\frac{1}{2})} &= (\alpha I - S(A) + i\eta I)x^{(k)} + b, \\
(\alpha I + S(A) - i\eta I)x^{(k+1)} &= (\alpha I - H(A) - i\eta I)x^{(k+\frac{1}{2})} + b.
\end{aligned}
\] (3.2)
If \( \eta = 0 \), the iteration scheme (3.2) reduces to the Hermitian and skew-Hermitian splitting (HSS) iteration method proposed in [16]. This implies that the HSS iteration method is a special case of the PPS iteration method.

In the following, we shall establish the convergence analysis of (3.2). For this purpose, we introduce the quantities
\[
\begin{aligned}
V(\alpha, \eta) &= (\alpha I - H(A) - i\eta I)(\alpha I + H(A) + i\eta I)^{-1}, \\
W(\alpha, \eta) &= (\alpha I - S(A) + i\eta I)(\alpha I + S(A) - i\eta I)^{-1}, \\
S(\alpha, \eta) &= (\alpha I + S(A) - i\eta I)^{-1}V(\alpha, \eta)(\alpha I - S(A) + i\eta I), \\
T(\alpha, \eta) &= 2\alpha[(\alpha I + H(A) + i\eta I)(\alpha I + S(A) - i\eta I)]^{-1}.
\end{aligned}
\] (3.3)
(3.4)
Then it is easy to verify that the iteration scheme (3.2) can be equivalent to
\[
x^{(k+1)} = S(\alpha, \eta)x^{(k)} + T(\alpha, \eta)b.
\] (3.5)
This shows that the method (3.2) is convergent if and only if the spectral radius of the iterative matrix \( S(\alpha, \eta) \) is less than 1.

**Theorem 3.1.** Let \( A \in \mathbb{C}^{n \times n} \) be a positive definite matrix, \( M \) and \( N \) be defined as in (3.1), \( \lambda_1 \) and \( \lambda_n \) be the maximum and minimum eigenvalues of \( H(A) \). Then for any \( \eta \in \mathbb{R} \) and \( \alpha > 0 \),
1. the sequence generated by the iteration scheme (3.2) converges to the exact solution \( x^* \in \mathbb{C}^n \) of the system (1.1);

2. the spectral radius \( \rho(S(\alpha, \eta)) \) of the iteration matrix \( S(\alpha, \eta) \) is bounded by

\[
\rho(S(\alpha, \eta)) \leq \sqrt{1 - \varphi(\alpha, \eta)},
\]

where

\[
\varphi(\alpha, \eta) = \min \left\{ \frac{4\alpha \lambda_1}{(\alpha + \lambda_1)^2 + \eta^2}, \frac{4\alpha \lambda_n}{(\alpha + \lambda_n)^2 + \eta^2} \right\}.
\]

Proof. Noticing that \( A \) and \( M \) are positive definite, and \( N \) is positive semi-definite, then by Theorem 2.1, we can get the first result. Now we estimate the bound of \( \rho(S(\alpha, \omega)) \).

It is easy to see that \((\alpha I - S(A) + \eta I)^{-1}S(A) = S(A)(\alpha I - S(A) + \eta I)^{-1}\). Then we can know that

\[
W(\alpha, \eta)^*W(\alpha, \eta) = (\alpha I - S(A) + \eta I)^{-1}((\alpha I - S(A) + \eta I)(\alpha I - S(A) + \eta I)^{-1}
\]

\[
= (\alpha I - S(A) + \eta I)(\alpha I - S(A) + \eta I)^{-1}((\alpha I - S(A) + \eta I)(\alpha I - S(A) + \eta I)^{-1}
\]

\[= I.
\]

Similarly, we can get \( W(\alpha, \eta)W(\alpha, \eta)^* = I \). This shows that \( W(\alpha, \eta) \) is a unitary matrix and \( \|W(\alpha, \eta)\|_2 = 1 \). This, along with the fact that \( S(\alpha, \eta) \) is similar to \( V(\alpha, \eta)W(\alpha, \eta) \), leads to

\[
\rho(S(\alpha, \eta)) = \rho(V(\alpha, \eta)W(\alpha, \eta)) \leq ||V(\alpha, \eta)||_2 ||W(\alpha, \eta)||_2 = ||V(\alpha, \eta)||_2.
\]

On the other hand, by Rayleigh-Ritz theorem, it follows that

\[
||V(\alpha, \eta)||_2^2 = \rho(V(\alpha, \eta)^*V(\alpha, \eta))
\]

\[
= \rho\left((\alpha I + H(A) + \eta I)^{-1}(\alpha I + H(A) + \eta I)(\alpha I + H(A) + \eta I)^{-1}\right)
\]

\[
= \max_{\lambda \in \Lambda(H(A))} \frac{(\alpha - \lambda + \eta)(\alpha - \lambda - \eta)}{(\alpha + \lambda + \eta)(\alpha + \lambda - \eta)} = \max_{\lambda \in \Lambda(H(A))} \frac{(\alpha - \lambda)^2 + \eta^2}{(\alpha + \lambda)^2 + \eta^2}.
\]

Noticing that for any \( \lambda \in \Lambda(H(A)) \), we can get

\[
\frac{(\alpha - \lambda)^2 + \eta^2}{(\alpha + \lambda)^2 + \eta^2} = \frac{\alpha^2 - 2\alpha \lambda + \lambda^2 + \eta^2}{\alpha^2 + 2\alpha \lambda + \lambda^2 + \eta^2} = \frac{\alpha^2 + 2\alpha \lambda + \lambda^2 + \eta^2 - 4\alpha \lambda}{\alpha^2 + 2\alpha \lambda + \lambda^2 + \eta^2}
\]

\[
= 1 - \frac{4\alpha \lambda}{\alpha^2 + 2\alpha \lambda + \lambda^2 + \eta^2} = 1 - \frac{4\alpha}{(\alpha^2 + \eta^2)/\lambda + \lambda + 2\alpha}.
\]

Since \( \alpha > 0 \) and \( \alpha^2 + \eta^2 > 0 \), we have

\[
\max_{\lambda \in \Lambda(H(A))} \frac{\alpha^2 + \eta^2}{\lambda} + \lambda = \max \left\{ \frac{\alpha^2 + \eta^2}{\lambda_1} + \lambda_1, \frac{\alpha^2 + \eta^2}{\lambda_n} + \lambda_n \right\}.
\]
This, together with \(\alpha > 0\), (3.8) and (3.9), yields that

\[
\|V(\alpha, \eta)\|^2_2 = \max_{\lambda \in \lambda(H(A))} \left[ 1 - \frac{4\alpha}{(\alpha^2 + \eta^2)/\lambda + \lambda + 2\alpha} \right]
\]

\[
= 1 - \min_{\lambda \in \lambda(H(A))} \frac{4\alpha}{(\alpha^2 + \eta^2)/\lambda + \lambda + 2\alpha}
\]

\[
= 1 - \min \left\{ \frac{4\alpha \lambda_1}{(\alpha^2 + \eta^2)/\lambda_1 + \lambda_1 + 2\alpha}, \frac{4\alpha \lambda_n}{(\alpha^2 + \eta^2)/\lambda_n + \lambda_n + 2\alpha} \right\}
\]

\[
= 1 - \min \left\{ \frac{4\alpha \lambda_1}{\lambda_1}, \frac{4\alpha \lambda_n}{\lambda_n} \right\}
\]

Then the second result follows. \(\square\)

From the above theorem, we can see that the optimal point \((\alpha^*, \eta^*)\) of \(\rho(S(\alpha, \eta))\) make \(\varphi(\alpha, \eta)\) maximum. This implies that we just need to solve the following optimization problem

\[
\max_{\alpha > 0, \eta \in \mathbb{R}} \varphi(\alpha, \eta)
\]

if we need to find the optimal parameters.

**Theorem 3.2.** Under the same settings and conditions as in Theorem 3.1, for any \(\eta \in \mathbb{R}\) and \(\alpha > 0\), it holds that

\[
(\alpha^*, \eta^*) \equiv \arg \max_{\alpha > 0, \eta \in \mathbb{R}} \varphi(\alpha, \eta) = (\sqrt{\lambda_1}, 0), \quad (3.10)
\]

\[
\varphi(\alpha^*, \eta^*) = \frac{4\sqrt{\lambda_1 \lambda_n}}{(\sqrt{\lambda_1} + \sqrt{\lambda_n})^2}. \quad (3.11)
\]

**Proof.** From (3.6b), we can see that the function \(\varphi(\alpha, \eta)\) decreases monotonically with respect to \(\eta^2\) for any \(\alpha > 0\). This shows that

\[
\eta^* \equiv \max_{\alpha > 0} \varphi(\alpha, \eta) = 0, \quad (3.12a)
\]

\[
\varphi(\alpha, 0) = \min \left\{ \frac{4\alpha \lambda_1}{(\alpha + \lambda_1)^2}, \frac{4\alpha \lambda_n}{(\alpha + \lambda_n)^2} \right\}. \quad (3.12b)
\]

In order to find the maximum point of \(\varphi(\alpha, 0)\), in the following, we derive \(\alpha^*\) such that minimize the function \(1/\varphi(\alpha, 0)\). It follows from (3.12b) that

\[
\frac{1}{\varphi(\alpha, 0)} = \max \left\{ \frac{(\alpha + \lambda_1)^2}{4\alpha \lambda_1}, \frac{(\alpha + \lambda_n)^2}{4\alpha \lambda_n} \right\}
\]

\[
= \max \left\{ \frac{\alpha}{4\alpha}, \frac{\lambda_1}{4\alpha}, \frac{1}{2}, \frac{\alpha}{4\lambda_n}, \frac{\lambda_n}{4\alpha}, \frac{1}{2} \right\}. \quad (3.13)
\]

Since \(H(A)\) is Hermitian positive definite, we have \(\lambda_1 > 0\) and \(\lambda_n > 0\). Then it is easy to see that \(\alpha^* = \sqrt{\lambda_1 \lambda_n}\). This, along with the independence of \(\alpha\) and \(\eta\), leads to (3.10) which immediately yields (3.11). This completes the proof. \(\square\)

**Remark 3.1.** We shall emphasize that \((\sqrt{\lambda_1 \lambda_n}, 0)\) is only the minimum point of \(\sqrt{1 - \varphi(\alpha, \eta)}\), rather than that of \(\rho(S(\alpha, \eta))\). Therefore, from Theorem 3.3, we can see that the optimal point \((\alpha^*, \eta^*)\) is the quasi-optimal parameter of \(\rho(S(\alpha, \eta))\), and the quasi-optimal spectral radius is

\[
\rho(S(\alpha^*, \eta^*)) \leq \sqrt{1 - \varphi(\alpha^*, \eta^*)} = \frac{\sqrt{\lambda_1} - \sqrt{\lambda_n}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}, \quad (3.14)
\]
In the following, we shall study another typical practical choice of the PPS method. Let

\[ A = D + L + U, \]  

(3.15)

where \( D, L \) and \( U \) are the diagonal, the strictly lower-triangular, and the strictly upper-triangular matrices of the matrix \( A \). For any \( \omega \in \mathbb{R} \), let

\[
M = D + L + U^* + i\omega I, \quad N = U - U^* - i\omega I, \quad (3.16a) \\
M = D + L^* + U + i\omega I, \quad N = L - L^* - i\omega I, \quad (3.16b) 
\]

it is easy to verify that \( A = M + N \). Noticing that

\[
M + M^* = D + L + U^* + i\omega I + D^* + L^* + U - i\omega I = A + A^*, \quad (3.17a) \\
M + M^* = D + L^* + U + i\omega I + D^* + L + U^* - i\omega I = A + A^*, \quad (3.17b) 
\]

we can know that the matrix \( M \) defined above is positive definite. On the other hand, from the fact \( N + N^* = 0 \), it follows that \( N \) is positive semi-definite. Then we can get the PPS iteration method as follows:

\[
\begin{cases}
(\alpha I + D + L + U^* + i\omega I)x^{(k+\frac{1}{2})} = (\alpha I - U + U^* + i\omega I)x^{(k)} + b, \\
(\alpha I + U - U^* - i\omega I)x^{(k+1)} = (\alpha I - D - L - U^* - i\omega I)x^{(k+\frac{1}{2})} + b,
\end{cases} \quad (3.18) 
\]

or

\[
\begin{cases}
(\alpha I + D + L^* + U + i\omega I)x^{(k+\frac{1}{2})} = (\alpha I - L + L^* + i\omega I)x^{(k)} + b, \\
(\alpha I + L - L^* - i\omega I)x^{(k+1)} = (\alpha I - D - L^* - U - i\omega I)x^{(k+\frac{1}{2})} + b.
\end{cases} \quad (3.19) 
\]

Particularly, if \( \omega = 0 \), then the above iteration formulas reduce to the triangular and skew-Hermitian splitting splitting (TSS) iteration method proposed in [14]. This shows that the TSS iteration method is a special case of the PPS iteration method. In addition, it is easy to see that the iterative formula (3.18) and (3.19) have same structure and property. Hence we just need to analyze the convergence of (3.18) in the following. Let

\[
V(\alpha, \omega) = (\alpha I - D - L - U^* - i\omega I)(\alpha I + D + L + U^* + i\omega I)^{-1}, \\
W(\alpha, \omega) = (\alpha I - U + U^* + i\omega I)(\alpha I + U - U^* - i\omega I)^{-1}, \\
S(\alpha, \omega) = (\alpha I + U - U^* - i\omega I)^{-1}V(\alpha, \omega)(\alpha I - U + U^* + i\omega I), \quad (3.20) \\
T(\alpha, \omega) = 2\alpha[(\alpha I + D + L + U^* + i\omega I)(\alpha I + U - U^* - i\omega I)]^{-1}. \quad (3.21) 
\]

Then the iteration scheme (3.18) can be reformulated equivalently as

\[
x^{(k+1)} = S(\alpha, \omega)x^{(k)} + T(\alpha, \omega)b. \quad (3.22) 
\]

Firstly, we propose the following lemmas.

**Lemma 3.1.** Let \( D, U, L \) be defined as in (3.15), and let \( t_1 = \alpha - i\omega, t_2 = \alpha + i\omega, Q = L + U^* \). Then it holds that

\[
V(\alpha, \omega) = (t_1 I - D)(t_2 I + D)^{-1} + 2\alpha(t_2 I + D)^{-1}\sum_{j=1}^{n-1}(-1)^j[Q(t_2 I + D)^{-1}]^j. \quad (3.23) 
\]
Theorem 3.3.

\[ ∥Q∥ = \text{that} \]

Using the definition of \( Q \), the Hamilton-Cayley theorem, yields \[ Q \]

\[ Q \]

Noticing that \( t \)

\[ N. HUANG AND C.F. MA \]

For any \( \alpha, \omega \)

\[ \text{Lemma 3.2.} \]

By the above analysis, we are now in a position to present the following theorem.

\[ \text{Lemma 3.2.} \]

For any \( \alpha > 0 \) and \( \omega \in R \), the matrix \( W(\alpha, \omega) \) is a unitary matrix, which follows that \( \| W(\alpha, \omega) \|_2 = 1 \).

By the above analysis, we are now in a position to present the following theorem.

Theorem 3.3.

Assume \( A \in \mathbb{C}^{n \times n} \) is a positive definite matrix. Let \( D = \text{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\} \), \( L \) and \( U \) be the diagonal, the strictly lower-triangular, and the strictly upper-triangular matrices of the matrix \( A \). Then for any \( \alpha > 0 \) and \( \omega \in R \),
(1) the sequence generated by the iteration scheme (3.18) converges to the exact solution \( x^* \in \mathbb{C}^n \) of the system (1.1);

(2) the spectral radius \( \rho(S(\alpha, \omega)) \) of the iteration matrix \( S(\alpha, \omega) \) is bounded by

\[
\rho(S(\alpha, \omega)) \leq \max_{1 \leq i \leq n} \frac{|\alpha - i\omega - a_{ii}|}{|\alpha + i\omega + a_{ii}|},
\]

approximately;

(3) If \( a_{ii}, i = 1, \ldots, n \) are all real with \( a_{\min} \) and \( a_{\max} \) being the minimum and the maximum elements of \( a_{ii}, i = 1, \ldots, n \), respectively, then it holds that

\[
(\alpha^*, \omega^*) \equiv \arg \min_{\alpha > 0, \omega \in \mathbb{R}} \psi(\alpha, \omega) = (\sqrt{a_{\min}}a_{\max}, 0),
\]

where

\[
\psi(\alpha, \omega) = \max_{1 \leq i \leq n} \frac{|\alpha - i\omega - a_{ii}|}{|\alpha + i\omega + a_{ii}|},
\]

Proof. As the iteration scheme (3.18) is a special case of the PPS iteration method, from Theorem 2.1, we get the first result.

Since \( A \) is a positive definite matrix, we have \( \text{Re}(a_{ii}) > 0, i = 1, \ldots, n \). It follows from Lemmas 3.1 and 3.2 that

\[
\rho(S(\alpha, \omega)) = \rho(V(\alpha, \omega)W(\alpha, \omega)) \leq \|V(\alpha, \omega)\|_2\|W(\alpha, \omega)\|_2 \\
\approx \|(t_1I - D)(t_2I + D)^{-1}\|_2 \\
= \max_{1 \leq i \leq n} \frac{|t_1 - a_{ii}|}{|t_2 + a_{ii}|} = \max_{1 \leq i \leq n} \frac{|\alpha - i\omega - a_{ii}|}{|\alpha + i\omega + a_{ii}|} < 1,
\]

where the last equality holds by the definition of \( t_1 \) and \( t_2 \). Then the second result follows.

Furthermore, if \( a_{ii}, i = 1, \ldots, n \) are all real, then \( a_{ii} > 0 \). This yields that

\[
\psi(\alpha, \omega) = \max_{1 \leq i \leq n} \frac{|\alpha - i\omega - a_{ii}|}{|\alpha + i\omega + a_{ii}|} = \max_{1 \leq i \leq n} \left[ \frac{(\alpha - a_{ii})^2 + \omega^2}{(\alpha + a_{ii})^2 + \omega^2} \right]^{\frac{1}{2}}.
\]

For any \( i = 1, \ldots, n \), since

\[
\frac{(\alpha - a_{ii})^2 + \omega^2}{(\alpha + a_{ii})^2 + \omega^2} = \frac{(\alpha + a_{ii})^2 + \omega^2 - 4\alpha a_{ii}}{(\alpha + a_{ii})^2 + \omega^2} = 1 - \frac{4\alpha a_{ii}}{(\alpha + a_{ii})^2 + \omega^2},
\]

we can know that \( \psi(\alpha, \omega) \) increases monotonically with respect \( \omega^2 \) for any \( \alpha > 0 \). Then

\[
\omega^* \equiv \arg \min_{\omega \in \mathbb{R}} \psi(\alpha, \omega) = 0,
\]

which immediately follows that

\[
\psi(\alpha, \omega^*) = \max_{1 \leq i \leq n} \frac{|\alpha - a_{ii}|}{|\alpha + a_{ii}|} = \max \left\{ \frac{|\alpha - \alpha_{\min}|}{|\alpha + \alpha_{\min}|}, \frac{|\alpha - \alpha_{\max}|}{|\alpha + \alpha_{\max}|} \right\}.
\]

If \( \alpha^* \) is a minimum point of \( \psi(\alpha, \omega^*) \), then it must satisfy that \( \alpha - \alpha_{\min} > 0, \alpha - \alpha_{\max} < 0 \), and

\[
\frac{\alpha - \alpha_{\min}}{\alpha + \alpha_{\min}} = \frac{\alpha_{\max} - \alpha}{\alpha + \alpha_{\max}}.
\]
By some simple calculation, we can get 

\[ \alpha^* \equiv \arg \min_{\alpha > 0} \psi(\alpha, \omega) = \sqrt{a_{\min} a_{\max}}. \]

This, along with the independence of \( \alpha \) and \( \omega \), leads to (3.28a) and (3.28b).

**Remark 3.2.** We shall emphasize that the condition of the third result on the diagonal elements is necessary. Otherwise, we cannot derive the specific formula of \( \psi(\alpha, \omega) \). For example, let 

\[ A = \begin{pmatrix} 1 + i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 - \frac{1}{2}i \end{pmatrix}. \]

Evidently, \( A \) is positive definite and \( a_{11} = 1 + i \), \( a_{22} = 2 \), \( a_{33} = 1 - \frac{1}{2}i \). Then for \( \alpha = 1 \) and \( \omega = 0 \), we have 

\[ \frac{|1 - a_{11}|}{|1 + a_{11}|} = \frac{1}{\sqrt{5}}, \quad \frac{|1 - a_{22}|}{|1 + a_{22}|} = \frac{1}{3}, \quad \frac{|1 - a_{33}|}{|1 + a_{33}|} = \frac{1}{\sqrt{17}}. \]

This shows that 

\[ \psi(1, 0) = \frac{|1 - a_{11}|}{|1 + a_{11}|}. \]

However, \( a_{11} \) does not have apparent characteristic point, such as maximum modulus or minimum modulus. Therefore, for general problems, we cannot determine the expression of \( \psi(\alpha, \omega) \). And then we cannot calculate the minimum of \( \psi(\alpha, \omega) \).

### 4. Numerical Experiments

In this section, we use two examples to numerically examine feasibility and effectiveness of our new methods. All our tests are started from zero vector, and terminated when the current iteration satisfies 

\[ \frac{\| r_k \|_2}{\| r_0 \|_2} < 10^{-5}, \quad (4.1) \]

where \( r_k \) is the residual of the current, say k-th, iteration. We report the number of required iterations (IT), the norm of the relative residual (4.1) (Res) when the process is stopped, the required CPU time (CPU). All algorithms were coded by MATLAB 2011b and were run on a PC with 2.20 GHz Pentium(R) Dual-Core CPU and 2.00 GB RAM.

**Example 4.1.** Consider the system of linear equations (1.1), for which 

\[ A = \text{tridiag}(-1 + i, 10, 1 - i), \quad b = (1, 1, \cdots, 1)^T. \]

Since \( A + A^* = \text{tridiag}(2i, 20, -2i) \), \( A \) is non-Hermitian positive definite. We split \( A \) into \( M + N \) with 

\[ M = \text{tridiag}(0, 5, 1 - i), \quad \text{and} \quad N = \text{tridiag}(-1 + i, 5, 0). \]

This shows that 

\[ M + M^* = \text{tridiag}(1 + i, 10, 1 - i), \quad \text{and} \quad N + N^* = \text{tridiag}(-1 + i, 10, -1 - i). \]
Table 4.1: Numerical results of Example 4.1.

<table>
<thead>
<tr>
<th>HSS</th>
<th>TSS</th>
<th>PPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>IT</td>
<td>CPU</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>14.0936</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>10.7445</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>8.8524</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>7.1883</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>5.6605</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>4.9500</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>3.4438</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>2.8982</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>3.5298</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>4.2154</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>4.7301</td>
</tr>
</tbody>
</table>

Fig. 4.1. $\rho(M(\alpha))$ versus $\alpha$ for the PPS, the HSS and the TSS iteration matrices for Example 4.1. Hence both $M$ and $N$ are positive definite. In our test, we compare the PPS method based on this splitting with the HSS method and the TSS method. We take the dimension $n = 1024$. The test results are listed in Table 4.1. And we depict the curves of $\rho(M(\alpha))$ with respect to $\alpha$ for all PPS, HSS and TSS iteration methods to intuitively show this functional relationship in Figure 4.1.

From Table 4.1, we can see that all the methods stopped regularly and the optimal parameter $\alpha_{opt}$ of the PPS method is in the interval of $[5, 6]$. The optimal parameters of the HSS method and the TSS method are both near by 10. This fact is further confirmed by Figure 4.1. As shown in Table 4.1, whatever the method, the number of iterations and the actual computing time vary seriously for the different parameter $\alpha$. Furthermore, whatever the parameter $\alpha$ we choose, the CPU time of the PPS method is always the least.
Table 4.2: Numerical results of Example 4.2.

<table>
<thead>
<tr>
<th>(\eta/\omega)</th>
<th>(\alpha_{\text{opt}})</th>
<th>IT</th>
<th>CPU</th>
<th>Res</th>
<th>(\alpha_{\text{opt}})</th>
<th>IT</th>
<th>CPU</th>
<th>Res</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6</td>
<td>12.9086</td>
<td>66</td>
<td>4.1304</td>
<td>8.2780e-006</td>
<td>12.4248</td>
<td>65</td>
<td>2.9226</td>
<td>8.1309e-006</td>
</tr>
<tr>
<td>0</td>
<td>14.0936</td>
<td>66</td>
<td>0.5630</td>
<td>7.5525e-006</td>
<td>13.8493</td>
<td>59</td>
<td>0.5602</td>
<td>6.8548e-006</td>
</tr>
</tbody>
</table>

Example 4.2. (see [14]) Consider the system of linear equations (1.1), for which

\[
A = \begin{pmatrix} W & \Omega \\ -F^T & N \end{pmatrix},
\]

where \(W \in \mathbb{R}^{q \times q}\), and \(N, \Omega \in \mathbb{R}^{(n-q) \times (n-q)}\), with \(2q > n\), \(b = Ae\), \(e = (1, \cdots, 1)^T\).

In particular, the matrices \(W, N, F\) and \(\Omega = \text{diag}\{\nu_1, \cdots, \nu_{n-q}\}\) are defined as follows:

\[
w_{kj} = \begin{cases} 
  k + 1, & \text{for } j = k, \\
  1, & \text{for } |k - j| = 1, \\
  0, & \text{otherwise},
\end{cases} \quad k, j = 1, \cdots, q,
\]

\[
n_{kj} = \begin{cases} 
  k + 1, & \text{for } j = k, \\
  1, & \text{for } |k - j| = 1, \\
  0, & \text{otherwise},
\end{cases} \quad k, j = 1, \cdots, n - q,
\]

\[
f_{kj} = \begin{cases} 
  j, & \text{for } k = j + 2q - n, \\
  0, & \text{otherwise},
\end{cases} \quad k = 1, \cdots, q, j = 1, \cdots, n - q,
\]

\[
\nu_k = \frac{1}{k} \quad k = 1, \cdots, n - q.
\]

In this example, we compare the iteration scheme (3.2) (HPPS for short) with the iteration scheme (3.19) (TPPS for short) with different parameter \(\eta\) or \(\omega\). In our computations, we choose \(n = 200\) and \(q = \frac{q}{n}n\). The test results are listed in Table 4.2. And we depict the surfaces of \(\rho(S(\alpha, \eta))\) (or \(\rho(S(\alpha, \omega))\)) with respect to \(\alpha\) and \(\eta\) (or \(\omega\)) for HPPS (or TPPS) iteration method to intuitively show this functional relationship in Figure 4.2 (or Figure 4.3). Moreover, we depict the eigenvalue distributions of the HPPS (or TPPS) iteration matrices for different parameter \(\eta\) (or \(\omega\)) and the corresponding optimal parameter \(\alpha_{\text{opt}}\) in Figure 4.4 (or Figure 4.5).

As can be seen from Table 4.2, when \(\eta = 0\), the number of iterations and the CPU time of the HPPS method are the best. And the HPPS method may be slowed by increasing the module of \(\eta\). Moreover, we can see that the numerical results of the HPPS method are much the same when the module of \(\eta\) is equal. And the optimal parameter \(\alpha_{\text{opt}}\) is equal when the module of \(\eta\) is equal. This fact is further confirmed by Figure 4.2. For TPPS method, we have the same results as the HPPS method, which can be seen from Table 4.2 and Figure 4.3. As is shown in Figure 4.4, the distribution domain of the eigenvalues of the HPPS iteration matrix
Fig. 4.2. Surfaces of $\rho(S(\alpha, \eta))$ versus $\alpha$ and $\eta$ for the HPPS iteration matrix for Example 4.2.

Fig. 4.3. Surfaces of $\rho(S(\alpha, \omega))$ versus $\alpha$ and $\omega$ for the TPPS iteration matrix for Example 4.2.

Fig. 4.4. The eigenvalue distributions of the HPPS iteration matrices when $\eta = -4, 0, 4$ for Example 4.2.
Fig. 4.5. The eigenvalue distributions of the TPPS iteration matrices when $\omega = -4, 0, 4$ for Example 4.2.

Fig. 4.6. Curves of $\rho(S(\alpha))$ versus $\alpha$ for the HPPS and the TPPS iteration matrices for Example 4.2 by $\eta = \omega = 0.25$.

when $\eta = 0$ is considerably smaller than that when $\eta \neq 0$, in particular, along the direction of the imaginary axis. From Figure 4.5, we can see the same result for the TPPS iteration matrix.

In addition, as you will see in Table 4.2, the numerical results of the TPPS method are a little better than that of the HPPS method, which also can be seen from Figure 4.6.

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References

Splitting Methods for Non-Hermitian Positive Definite Linear Systems


