LOCAL STRUCTURE-PRESERVING ALGORITHMS FOR THE KDV EQUATION

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Abstract

In this paper, based on the concatenating method, we present a unified framework to construct a series of local structure-preserving algorithms for the Korteweg-de Vries (KdV) equation, including eight multi-symplectic algorithms, eight local energy-conserving algorithms and eight local momentum-conserving algorithms. Among these algorithms, some have been discussed and widely used while the most are new. The outstanding advantage of these proposed algorithms is that they conserve the local structures in any time-space region exactly. Therefore, the local structure-preserving algorithms overcome the restriction of global structure-preserving algorithms on the boundary conditions. Numerical experiments are conducted to show the performance of the proposed methods. Moreover, the unified framework can be easily applied to many other equations.


Key words: Korteweg-de Vries (KdV) equation, structure-preserving algorithms, concatenating method, multi-symplectic conservation law.

1. Introduction

Consider the system of Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \eta u \frac{\partial u}{\partial x} + \mu^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, \tag{1.1}$$

where $\eta$ and $\mu$ are both real constants. It is an important nonlinear hyperbolic equation with smooth solutions at all times and also a mathematical model of waves on shallow water surfaces [1]. Eq. (1.1) has been used to describe various kinds of phenomena, such as waves in bubble-liquid mixtures, acoustic waves in an anharmonic crystal, magnetohydrodynamic wave in warm plasmas and ion acoustic wave. The KdV equation was originally introduced by Zabusky-Kruskal [2] who discovered the soliton in 1965.

Based on the rule that numerical algorithms should preserve the intrinsic properties of the original problems as much as possible, Feng [3] first presented the concept of symplectic schemes for Hamiltonian systems and further the structure-preserving algorithms for the general conservative dynamical systems. Theoretical analysis and practical computations both prove that
the symplectic schemes have very wide and significant applications in many fields due to their excellent stability and accurate long time simulations [4–6]. Marsden et al. [7], Bridges [8] and Reich [9] introduced the concept of multi-symplectic structure and multi-symplectic integrator for Hamiltonian partial differential equations (PDEs), which can be regarded as the direct generalization of symplectic integrator. Afterwards, multi-symplectic algorithms developed very fast and a lot of achievements have been obtained. For example, Hong et al. [10] and Liu et al. [11] proposed the multi-symplectic Runge-Kutta methods for nonlinear Dirac equations and Hamiltonian equations, respectively. Sun et al. [12] and Kong et al. [13] investigated the multi-symplectic methods for Maxwell equation. Moreover, Hong et al. [14] studied the multi-symplecticity of partitioned Runge-Kutta methods for Hamiltonian PDEs. Wang and Hong [15] reviewed the development of multi-symplectic algorithms for Hamiltonian PDEs. Actually, besides the geometric structure, the idea of Feng’s structure-preserving algorithms also contains other conservative properties of the PDEs, such as the physical conservation laws like energy or momentum conservation law and the algebraic characters. In some fields, it is convenient sometimes to construct numerical algorithms that preserve the physical conservation law rather than the symplectic or multi-symplectic ones.

It is noted that Wang et al. [16] proposed the concept of the local structure-preserving algorithm for PDEs, and then constructed some algorithms preserved the multi-symplectic conservation law, local energy and momentum conservation laws for the Klein-Gordon equation by using the concatenating method. Cai et al. [17, 18] applied successfully the theory of the local structure-preserving algorithm to the “good” Boussinesq equation and the coupled nonlinear Schrödinger system. The main advantage of local structure-preserving algorithm is that they conserve the local structures of PDEs in any local time-space region. In other words, they can overcome the restriction of global structure-preserving algorithm on the boundary conditions.

On the other hand, there have been many numerical methods for solving the KdV equation, such as finite difference methods, finite element methods, spectral and pseudo-spectral methods. Note that structure-preserving algorithms play an important role in the development of these numerical methods. In particular, a 12-point multi-symplectic scheme for KdV equation was derived by Zhao and Qin [19], and a family of symplectic and multi-symplectic box schemes of KdV equation was investigated by Ascher and McLachlan [20]. Wang et al. [21] discussed an explicit 6-point multi-symplectic scheme which did not show the nonlinear instabilities and unphysical oscillations when used to simulate the collision of multiple solitary wave. In [22], Chen et al. constructed a multi-symplectic Fourier pseudo-spectral method and a multi-symplectic wavelet collocation method for the Ito-type coupled KdV equation. In [23], Cui et al. developed a finite-volume scheme for the KdV equation, which conserved both the momentum and energy using the operator splitting approach. However, the above existing algorithms are put forward and studied just individually and except [23], there is few energy-conserving or momentum-conserving methods in literatures, which motivates us to study the local structure-preserving algorithms of the KdV equation systematically.

In this paper, we emphasize how to construct a unified framework of the local structure-preserving algorithms for KdV equation by using the concatenating method. This method to construct difference schemes for PDEs is different from the method of lines and the alternating direction method. Its basis idea comes from the Runge-Kutta method which deals with PDEs by space and time separately. In 2000, Reich [24] proved that concatenating the Runge-Kutta method in type Gauss collocation can lead to multi-symplectic schemes for the nonlinear wave equation. In [16–18, 25], Wang et al. used this method to construct a series of multi-symplectic
schemes for some Hamiltonian wave equations successfully. The local structure-preserving algorithms we proposed include eight multi-symplectic algorithms, eight local energy-conserving algorithms and eight local momentum-conserving algorithms. Among these algorithms, some are the schemes mentioned above while the most are novel. Furthermore, the discrete conservation law of each scheme and its proof are given. This framework can be easily applied to many other equations. We also provide the linear stability analysis and numerical experiments to verify the good numerical performance at last.

This paper is organized as follows. Some operator definitions and their properties are given in Section 2. In Section 3, we use the concatenating method to construct the local structure-preserving algorithms. Stability analysis is provided in Section 4. In Section 5, some numerical experiments are made to exhibit the good performance of the proposed algorithms. We finish this paper with some conclusions in Section 6.

2. Operator Definitions and Properties

We first introduce some notations: \( x_j = x_L + j \Delta x \), \( t_k = k \Delta t \), \( j = 1, \ldots, N+1 \), \( k = 0, 1, \ldots \), where \( \Delta x = (x_R - x_L) / N \), \( \Delta t \) are spatial length and temporal step span, respectively. The approximation of the value of the function \( u(x,t) \) at the node \((x_j, t_k)\) is denoted by \( u^j_k \). In order to derive the structure-preserving schemes conveniently, we also define the finite difference operators

\[
\delta_t f^j_i = \frac{f^j_{i+1} - f^j_i}{\Delta t}, \quad \delta_x f^j_i = \frac{f^j_i - f^j_{i-1}}{\Delta x},
\]

and averaging operators

\[
A_t f^j_i = \frac{f^j_i + f^j_{i+1}}{2}, \quad A_x f^j_i = \frac{f^j_i + f^j_{i-1}}{2}.
\]

The operators have the following properties:

- **Commutative law**
  \[
  \delta_x \delta_t f^j_i = \delta_t \delta_x f^j_i, \quad A_x A_t f^j_i = A_t A_x f^j_i, \quad \delta_x A_t f^j_i = A_t \delta_x f^j_i, \quad A_x \delta_t f^j_i = \delta_t A_x f^j_i.
  \]

- **Chain rule**
  \[
  \delta_x V(u^j_i) = \delta_u V(u^j_i) \delta_x u^j_i + o(\Delta t), \quad \delta_x V(u^j_i) = \delta_u V(u^j_i) \delta_x u^j_i + o(\Delta x).
  \]

- **Discrete Leibnitz rule**
  \[
  \delta_x (f \cdot g)^j_i = (af^j_{i+1} + (1-a)f^j_i) \cdot \delta_x g^j_i + \delta_x f^j_i \cdot ((1-a)g^j_{i+1} + ag^j_i), \quad \forall \ 0 \leq a \leq 1.
  \]

Especially, we have

- \( a = 0 \), \( \delta_x (f \cdot g)^j_i = f^j_i \cdot \delta_x g^j_i + \delta_x f^j_i \cdot g^j_{i+1} \),
- \( a = \frac{1}{2} \), \( \delta_x (f \cdot g)^j_i = A_x f^j_i \cdot \delta_x g^j_i + \delta_x f^j_i \cdot A_x g^j_i \),
- \( a = 1 \), \( \delta_x (f \cdot g)^j_i = f^j_{i+1} \cdot \delta_x g^j_i + \delta_x f^j_i \cdot g^j_i \).
Taking \( f = g \) gives
\[
\delta_x \left[ \frac{1}{2} (f^j_i)^2 \right] = \delta_x f^j_i \cdot A_x f^j_i, \quad \delta_x \left( \frac{1}{2} f^j_i \cdot f^j_{i-1} \right) = f^j_i \cdot A_x \delta_x f^j_{i-1}.
\]
Similarly, we can obtain a series of analogous discrete Leibnitz rules in the time direction. The Leibnitz rules play an important role in proving the local conservation law of the algorithm.

3. Concatenating Construction of the Algorithms

Let us review the properties of the KdV equation before the construction of local structure-preserving algorithms. By introducing the potential \( \varphi_x = u \), momenta \( v = \mu u_x \) and variable \( w = \frac{1}{2} \varphi + \mu v + \frac{1}{6} \eta u^3 \), the KdV equation (1.1) can be reformulated as the following first-order system
\[
\begin{cases}
\varphi_t = -2 \mu v_x + 2w - \eta u^2, \\
u_t = -2 \mu w_x, \\
\varphi_x = u, \quad u_x = \frac{1}{\mu} v.
\end{cases}
\]

This system can be written as a multi-symplectic system
\[
M \partial_x + K \partial_z = \nabla_x S(z), \quad z \in \mathbb{R}^4, \ (x, t) \in \mathbb{R}^2,
\]
where
\[
z = \begin{pmatrix} \varphi \\ u \\ v \\ w \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]
and Hamiltonian \( S(z) = \frac{1}{2} v^2 - uw + \frac{1}{6} \mu \eta u^3 \). The corresponding multi-symplectic, local energy and momentum conservation laws are
\[
\frac{\partial}{\partial t} (d\varphi \wedge du) + \frac{\partial}{\partial x} (2 d\varphi \wedge dw + 2 \mu dv \wedge du) = 0,
\]
\[
\frac{\partial}{\partial t} \left( -\frac{1}{2} v^2 + \frac{1}{6} \mu \eta u^3 \right) + \frac{\partial}{\partial x} (-w\varphi_t + \mu vu_t) = 0,
\]
\[
\frac{\partial}{\partial t} \left( -\frac{1}{2} u^2 + \frac{1}{6} \eta u^3 \right) + \frac{\partial}{\partial x} \left( \frac{1}{2} v^2 - uw + \frac{1}{6} \mu \eta u^3 + \frac{1}{2} \varphi_t \right) = 0.
\]

Under periodic boundary conditions, the above local conservation laws can be integrated in \( x \)-direction respectively to obtain the global symplectic, energy and momentum conservation laws
\[
W(t) = \int_{x_L}^{x_R} d\varphi \wedge du \ dx = \text{Constant},
\]
\[
E(t) = \int_{x_L}^{x_R} \left( -\frac{1}{2} v^2 + \frac{1}{6} \eta u^3 \right) \ dx = \text{Constant},
\]
\[
M(t) = -\frac{1}{2} \int_{x_L}^{x_R} u^2 \ dx = \text{Constant}.
\]

An algorithm is called a symplectic (multi-symplectic) algorithm for (1.1) or (3.1) if it can preserve the discrete form of the symplectic conservation law (3.5)(multi-symplectic conservation law).
law (3.2)). An algorithm is called an energy-conserving (local energy-conserving) algorithm for (1.1) or (3.1) if it can preserve the discrete form of the energy conservation law (3.6) (local energy conservation law (3.3)). Similarly, we can get the definition of a (local) momentum-conserving algorithm.

In this section, we construct the local structure-preserving algorithms systematically for the KdV equation by using the concatenating method, which is different from the method of lines and the alternating direction method. Its basic idea is from the Runge-Kutta method which deals with PDEs by time and space separately. By introducing the auxiliary variables \( v_x = a, \ w_x = b, \) the PDE system (1.1) can be written as a system of three simultaneous ordinary differential equations (ODEs)

\[
\begin{align*}
  v_x &= a, \quad u_x = \frac{1}{\mu} v_t, \\
  w_x &= b, \quad \varphi_t = u_t, \\
  \varphi_t &= -2\mu a + 2w - 2V'(u), \quad u_t = -2b,
\end{align*}
\]

where \( V(u) = \frac{1}{6} \eta u^3. \)

Discreting the three ODEs (3.8a)-(3.8c), then we can obtain the numerical schemes for the KdV equation by combining the three discreted ODEs. It is called the concatenating method in the sense that it concatenates different numerical schemes of ODEs to derive the structure-preserving scheme for PDEs. Here we apply Leap-frog rule and mid-point rule to Eqs. (3.8a)-(3.8c) respectively and then eliminate the introduced variables by combining the three discrete ODEs. We should note why we can get structure-preserving algorithms in this way. At first, the Leap-frog rule and mid-point rule we used here are both symplectic method and numerous literatures [16–18, 24] have shown that the concatenation of symplectic method can obtain the structure-preserving schemes successfully though there is no rigorous theoretical proof. Secondly, the discrete Leibnitz rules play a significant role in proving the local conservation law of the proposed algorithms. Last, we use the discrete gradient method in different directions to discretize the nonlinear term in the KdV equation for getting the energy-conserving and momentum-conserving algorithms, respectively.

3.1. Construction of multi-symplectic algorithms

First, applying Leap-frog rule and mid-point rule to the KdV equation in the spatial and temporal direction respectively, we construct eight multi-symplectic schemes.

3.1.1. Multi-symplectic scheme I (MS I)

Applying the mid-point rule to ODEs (3.8b)-(3.8c) and Leap-frog rule to ODE (3.8a), we have

\[
\begin{align*}
  \delta_x v^j_i &= a^j_i, \quad \delta_x u^j_i = \frac{1}{\mu} v_{i+1}^j, \\
  \delta_x w^j_i &= A_x b^j_i, \quad \delta_x \varphi^j_i = A_x u^j_i, \\
  \delta_t A_x \varphi^j_i &= -2\mu A_x A_t a^j_i + 2A_x A_t w^j_i - 2V'(A_x A_t u^j_i), \\
  \delta_t A_x u^j_i &= -2A_x A_t b^j_i
\end{align*}
\]
Eliminating the auxiliary variables \( a \) and \( b \) yields
\[
\begin{aligned}
\delta_t A_x \varphi^j_i &= -2 \mu \delta_x A_x A_t u^j_i + 2 A_x A_t u^j_i - 2 V'(A_x A_t u^j_i), \\
\delta_t A_x u^j_i &= -2 \delta_x A_t u^j_i, \\
\delta_x u^j_i &= \frac{1}{\mu} v^j_{i+1}, \quad \delta_x \varphi^j_i = A_x u^j_i.
\end{aligned}
\] (3.10)

Eliminating \( v, w \) and \( \varphi \) yields an equivalent scheme
\[
\delta_t A_x^2 u^j_i + \mu^2 \delta_x^3 A_x A_t u^j_{i-1} + \frac{1}{2} \eta \delta_x (A_x A_t u^j_i)^2 = 0.
\]

**Theorem 3.1.** **Scheme** (3.10) **is a multi-symplectic scheme, which has the following discrete multi-symplectic conservation law**
\[
\delta_t (A_x \varphi^j_i \wedge A_x du^j_i) + \delta_x (2A_x \varphi^j_i \wedge A_t du^j_i + 2 \mu A_x A_t du^j_i \wedge A_x A_t du^j_i) = 0.
\] (3.11)

**Proof.** The variational equation associated of Eq. (3.10) is
\[
\begin{aligned}
\delta_t A_x \varphi^j_i &= -2 \mu \delta_x A_x A_t \varphi^j_i + 2 A_x A_t \varphi^j_i - 2 V'(A_x A_t \varphi^j_i)A_x A_t du^j_i, \\
\delta_t A_x du^j_i &= -2 \delta_x A_t du^j_i, \\
\delta_x du^j_i &= \frac{1}{\mu} dv^j_{i+1}, \quad \delta_x \varphi^j_i = A_x du^j_i.
\end{aligned}
\] (3.12)

Taking the wedge product of the first equation of Eq. (3.12) with \( A_x A_t du^j_i \) yields
\[
\delta_t A_x \varphi^j_i \wedge A_x A_t du^j_i = -2 \mu \delta_x A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + 2 A_x A_t \varphi^j_i \wedge A_x A_t du^j_i \\
= -2 \mu \delta_x A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + 2 A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + \delta_x A_t \varphi^j_i.
\]

Taking the wedge product of the second equation of Eq. (3.12) with \( A_x A_t \varphi^j_i \) yields
\[
\delta_t A_x du^j_i \wedge A_x A_t \varphi^j_i = -2 \delta_x A_t du^j_i \wedge A_x A_t \varphi^j_i.
\]

Combining the above two equations yields
\[
\begin{aligned}
&\delta_t (A_x \varphi^j_i \wedge A_x du^j_i) = \delta_t (A_x \varphi^j_i \wedge A_x A_t du^j_i) - \delta_t (A_x du^j_i \wedge A_x A_t \varphi^j_i) \\
= -2 \mu \delta_x A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + 2 A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + \delta_x A_t \varphi^j_i + 2 \delta_x A_t du^j_i \wedge A_x A_t \varphi^j_i \\
= -2 \mu \delta_x (A_x A_t \varphi^j_i \wedge A_x A_t du^j_i) + 2 \mu A_x A_t du^j_i \wedge \delta_x A_x A_t du^j_i - 2 \delta_x (A_t \varphi^j_i \wedge A_t du^j_i) \\
= -\delta_x (2 \mu A_x A_t \varphi^j_i \wedge A_x A_t du^j_i + 2 A_t \varphi^j_i \wedge A_t du^j_i),
\end{aligned}
\]

which is the multi-symplectic conservation law (3.11). \( \square \)

### 3.1.2. Multi-symplectic scheme II (MS II)

Applying the Leap-frog rule to space derivatives and mid-point rule to time derivatives in ODEs (3.8a)-(3.8c), we have
\[
\begin{aligned}
\delta_x v^j_i &= v^j_i, \quad \delta_x u^j_i &= \frac{1}{\mu} v^j_{i+1}, \\
\delta_x w^j_i &= b^j_i, \quad \delta_x \varphi^j_i &= u^j_{i+1}, \\
\delta_x \varphi^j_i &= -2 \mu A_x b^j_i + 2 A_t w^j_i - 2 V'(A_t u^j_i), \\
\delta_x u^j_i &= -2 A_t b^j_i.
\end{aligned}
\] (3.13a-c)
Eliminating the auxiliary variables \(a\) and \(b\) yields
\[
\begin{align*}
\delta_t \phi_i^j &= -2\mu \delta_x A_t v_i^j + 2A_t w_i^j - 2V'(A_t w_i^j), \\
\delta_t w_i^j &= -\delta_x A_t v_i^j, \\
\delta_x v_i^j &= \frac{1}{\mu} v_i^{j+1}, \quad \delta_x \phi_i^j = w_i^{j+1}.
\end{align*}
\] (3.14)

Eliminating \(v, w\) and \(\phi\) yields an equivalent scheme
\[
\delta_t A_x u_i^j + \mu^2 \delta_x^3 A_t u_{i-1}^j + \frac{1}{2} \eta \delta_x (A_t u_i^j)^2 = 0.
\]

**Theorem 3.2.** Scheme (3.14) is a multi-symplectic scheme, which has the following discrete multi-symplectic conservation law
\[
\delta_t (d\phi_i^j \wedge du_i^j) + \delta_x (2A_t d\phi_{i-1}^j \wedge A_t du_i^j + 2\mu A_t dw_i^j \wedge A_t du_i^j) = 0.
\] (3.15)

**Proof.** As the proof of Theorem 3.1, the variational equation associated of Eq. (3.14) is
\[
\begin{align*}
\delta_t d\phi_i^j &= -2\mu \delta_x A_t v_i^j + 2A_t w_i^j - 2V'(A_t w_i^j)A_t du_i^j, \\
\delta_t du_i^j &= -2\delta_x A_t dw_i^j, \\
\delta_x du_i^j &= \frac{1}{\mu} du_i^{j+1}, \quad \delta_x d\phi_i^j = du_i^{j+1}.
\end{align*}
\] (3.16)

Taking the wedge product of the first equation of Eq. (3.16) with \(A_t du_i^j\) yields
\[
\delta_t d\phi_i^j \wedge A_t du_i^j = -2\mu \delta_x A_t v_i^j \wedge A_t dw_i^j + 2A_t dw_i^j \wedge A_t du_i^j
\] 
\[= -2\mu \delta_x A_t v_i^j \wedge A_t dw_i^j - 2\delta_x A_t dw_i^j \wedge A_t du_i^j.
\]

Taking the wedge product of the second equation of Eq. (3.16) with \(A_t d\phi_i^j\) yields
\[
\delta_t du_i^j \wedge A_t d\phi_i^j = -2\delta_x A_t dw_i^j \wedge A_t d\phi_i^j.
\]

Combining the above two equations yields
\[
\delta_t (d\phi_i^j \wedge du_i^j) = \delta_t d\phi_i^j \wedge A_t du_i^j - \delta_t du_i^j \wedge A_t d\phi_i^j
\]
\[= -2\mu \delta_x A_t v_i^j \wedge A_t dw_i^j - 2\delta_x A_t dw_i^j \wedge A_t du_i^j + 2\delta_x A_t dw_i^j \wedge A_t d\phi_i^j
\]
\[= -2\delta_x A_t dw_i^j \wedge A_t du_i^j + 2\delta_x A_t dw_i^j \wedge A_t d\phi_i^j
\]
\[= -\delta_x (2\mu A_t v_i^j \wedge A_t dw_i^j + 2A_t dw_i^j \wedge A_t du_i^j),
\]
which is the multi-symplectic conservation law (3.15).

**3.1.3. Multi-symplectic scheme III (MS III)**

Applying the mid-point rule to ODEs (3.8a), (3.8c) and Leap-frog rule to ODE (3.8b), we have
\[
\begin{align*}
\delta_x v_i^j &= A_x v_i^j, \quad \delta_x v_i^j = \frac{1}{\mu} A_x v_i^j, \quad \delta_x v_i^j = \frac{1}{\mu} A_x v_i^j, \\
\delta_x w_i^j &= b_i^j, \quad \delta_x \phi_i^j = u_i^{j+1}, \quad \delta_x \phi_i^j = u_i^{j+1}, \\
\delta_t A_x v_i^j &= -2\mu A_x A_t v_i^j + 2A_x A_t w_i^j - 2V'(A_t A_x u_i^j), \\
\delta_t A_x u_i^j &= -2A_x A_t b_i^j.
\end{align*}
\] (3.17)
Eliminating the auxiliary variables \(a\) and \(b\) yields
\[
\begin{aligned}
\delta_t A_x \varphi^l_i &= -2\mu \delta_x A_x v^l_i + 2A_x A_t w^l_i - 2V'(A_x A_t u^l_i), \\
\delta_t A_x u^l_i &= -2\delta_x A_x A_t w^l_i, \\
\delta_x u^l_i &= \frac{1}{\mu} A_x v^l_i, \quad \delta_x \varphi^l_i = u^l_{i+1}.
\end{aligned}
\tag{3.18}
\]

Eliminating \(v, w\) and \(\varphi\) yields an equivalent scheme
\[
\delta_t A_x^3 u^l_i + \mu^2 \delta_x^3 A_x u^l_i + \frac{1}{2\mu} \delta_x A_x (A_x A_t u^l_i)^2 = 0.
\]

**Theorem 3.3.** Scheme (3.18) is a multi-symplectic scheme, which has the following discrete multi-symplectic conservation law
\[
\delta_t (A_x d\varphi^l_i \wedge A_x d u^l_i) + \delta_x (2A_x A_t d\varphi^l_{i-1} \wedge A_x A_t d u^l_i + 2\mu A_t d u^l_i \wedge A_t d u^l_i) = 0.
\tag{3.19}
\]

**Proof.** Similar to the proof of previous theorems, the variational equation associated of Eq. (3.18) is
\[
\begin{aligned}
\delta_t A_x d\varphi^l_i &= -2\mu \delta_x A_x d u^l_i + 2A_x A_t d u^l_i - 2V'(A_x A_t u^l_i) A_x A_t d u^l_i, \\
\delta_t A_x d u^l_i &= -2\delta_x A_x A_t d u^l_i, \\
\delta_x d u^l_i &= \frac{1}{\mu} A_x d u^l_i, \quad \delta_x d\varphi^l_i = d u^l_{i+1}.
\end{aligned}
\tag{3.20}
\]

Taking the wedge product of the first equation of Eq. (3.20) with \(A_x A_t d u^l_i\) yields
\[
\delta_t A_x d\varphi^l_i \wedge A_x A_t d u^l_i = -2\mu \delta_x A_x d u^l_i \wedge A_x A_t d u^l_i + 2A_x A_t d u^l_i \wedge A_x A_t d u^l_i \\
-2\delta_x A_x A_t d u^l_i \wedge A_x A_t d u^l_i + 2A_x A_t d u^l_i \wedge \delta_x A_x A_t d\varphi^l_{i-1}.
\]

Taking the wedge product of the second equation of Eq. (3.20) with \(A_x A_t d\varphi^l_i\) yields
\[
\delta_t A_x d u^l_i \wedge A_x A_t d\varphi^l_i = -2\delta_x A_x A_t d u^l_i \wedge A_x A_t d\varphi^l_i.
\]

Combining the above two equations yields
\[
\begin{aligned}
\delta_t (A_x d\varphi^l_i \wedge A_x d u^l_i) &= \delta_t A_x d\varphi^l_i \wedge A_x d u^l_i - \delta_t A_x d u^l_i \wedge A_x A_t d\varphi^l_i \\
&= -2\mu \delta_x A_x d u^l_i \wedge A_x A_t d u^l_i + 2A_x A_t d u^l_i \wedge \delta_x A_x A_t d\varphi^l_{i-1} + 2\delta_x A_x A_t d u^l_i \wedge A_x A_t d\varphi^l_i \\
&= -2\mu \delta_x (A_t d u^l_i \wedge A_t d\varphi^l_i) + 2\mu A_t d u^l_i \wedge \delta_x A_t d u^l_i - 2\delta_x (A_x A_t d\varphi^l_{i-1} \wedge A_x A_t d u^l_i) \\
&= -\delta_x (2\mu A_t d u^l_i \wedge A_t d\varphi^l_i + 2A_x A_t d\varphi^l_{i-1} \wedge A_x A_t d u^l_i),
\end{aligned}
\]

which is the multi-symplectic conservation law (3.19). \(\square\)

### 3.1.4. Multi-symplectic scheme IV (MS IV)

Applying the mid-point rule to space derivatives as well as time derivatives in ODEs (3.8a)-(3.8c), we have
\[
\begin{aligned}
\delta_x v^l_i &= A_x a^l_i, \quad \delta_x u^l_i = \frac{1}{\mu} A_x v^l_i, \\
\delta_x w^l_i &= A_x b^l_i, \quad \delta_x \varphi^l_i = A_x u^l_i, \\
\delta_t A_x \varphi^l_i &= -2\mu A_x A_t a^l_i + 2A_x A_t w^l_i - 2V'(A_x A_t u^l_i), \\
\delta_t A_x u^l_i &= -2A_x A_t b^l_i.
\end{aligned}
\tag{3.21}
\]
Eliminating the auxiliary variables $a$ and $b$ yields

\[
\begin{align*}
\delta_t A_x \phi_i^j &= -2\mu \delta_x A_t v_i^j + 2A_x A_t w_i^j - 2V'(A_x A_t u_i^j), \\
\delta_t A_x w_i^j &= -2\delta_x A_t u_i^j, \\
\delta_x u_i^j &= \frac{1}{\mu} A_x v_i^j, \quad \delta_x \phi_i^j = A_x u_i^j.
\end{align*}
\] (3.22)

Eliminating $v$, $w$ and $\phi$ yields an equivalent scheme

\[
\delta_t A_x^3 u_i^j + \mu^2 \delta_x^2 A_t u_i^j + \frac{1}{2} \mu \delta_x A_x (A_x A_t u_i^j)^2 = 0.
\]

**Remark 3.1.** Scheme (3.22) is the multi-symplectic Preissmann scheme for the KdV equation, which has been discussed and verified to give some good numerical results on solitary waves over long-time intervals in [19].

**Theorem 3.4.** Scheme (3.22) is a multi-symplectic scheme, which has the following discrete multi-symplectic conservation law

\[
\delta_t (A_x d\phi_i^j \wedge A_x d\phi_i^j) + \delta_x (2A_t d\phi_i^j \wedge A_t d\phi_i^j + 2\mu A_t d\phi_i^j \wedge A_t d\phi_i^j) = 0. \tag{3.23}
\]

**Proof.** As the proof for previous theorems, the variational equation associated with Eq. (3.22) is

\[
\begin{align*}
\delta_t A_x d\phi_i^j &= -2\mu \delta_x A_t v_i^j + 2A_x A_t w_i^j - 2V'(A_x A_t u_i^j)A_x A_t du_i^j, \\
\delta_t A_x d\phi_i^j &= -2\delta_x A_t u_i^j, \\
\delta_x d\phi_i^j &= \frac{1}{\mu} A_x d\phi_i^j, \quad \delta_x d\phi_i^j = A_x d\phi_i^j.
\end{align*}
\] (3.24)

Taking the wedge product of the first equation of Eq. (3.24) with $A_x A_t d\phi_i^j$ yields

\[
\delta_t A_x d\phi_i^j \wedge A_x A_t d\phi_i^j = -2\mu \delta_x A_t v_i^j \wedge A_x A_t d\phi_i^j + 2A_x A_t w_i^j \wedge A_x A_t d\phi_i^j
\]

\[
= -2\mu \delta_x A_t v_i^j \wedge A_x A_t d\phi_i^j + 2A_x A_t w_i^j \wedge A_x A_t d\phi_i^j
\]

Taking the wedge product of the second equation of Eq. (3.24) with $A_x A_t d\phi_i^j$ yields

\[
\delta_t A_x d\phi_i^j \wedge A_x A_t d\phi_i^j = -2\delta_x A_t d\phi_i^j \wedge A_x A_t d\phi_i^j.
\]

Combining the above two equations yields

\[
\delta_t (A_x d\phi_i^j \wedge A_x d\phi_i^j) = \delta_t A_x d\phi_i^j \wedge A_x A_t d\phi_i^j - \delta_t A_x d\phi_i^j \wedge A_x A_t d\phi_i^j
\]

\[
= -2\mu \delta_x A_t v_i^j \wedge A_x A_t d\phi_i^j + 2A_x A_t w_i^j \wedge A_x A_t d\phi_i^j + 2\delta_x A_t d\phi_i^j \wedge A_x A_t d\phi_i^j
\]

\[
= -2\mu \delta_x (A_t d\phi_i^j \wedge A_t d\phi_i^j) + 2\mu A_x A_t d\phi_i^j \wedge A_x A_t d\phi_i^j - 2\delta_x (A_t d\phi_i^j \wedge A_t d\phi_i^j)
\]

\[
= -\delta_x (2\mu A_t d\phi_i^j \wedge A_t d\phi_i^j + 2A_t d\phi_i^j \wedge A_t d\phi_i^j),
\]

which is the multi-symplectic conservation law (3.23). \qed
3.1.5. Multi-symplectic scheme V-VIII (MS V-MS VIII)

We can obtain the other four multi-symplectic schemes in a similar manner and here we do not list the derivation process concretely due to the limited space. Applying the mid-point rule to ODE (3.8a) and Leap-frog rule to ODEs (3.8b)-(3.8c), we get

\[ \delta_t A_x^{i-1} u_{i+1}^{j} + \delta_t A_x^{i} u_{i}^{j-1} + 2\mu^2 \delta_x^2 u_{i}^{j} + \eta \delta_x (A_x u_{i}^{j})^2 = 0, \]

which has the following discrete multi-symplectic conservation law

\[ \delta_t (A_x d \phi_{i}^{j} \wedge A_x du_{i}^{j-1}) + \delta_x (2A_x d \phi_{i-1}^{j} \wedge A_x dw_{i}^{j} + 2\mu dv_{i}^{j} \wedge du_{i}^{j}) = 0. \]

Applying the mid-point rule to ODE (3.8b) and Leap-frog rule to ODEs (3.8a) and (3.8c), we get

\[ \delta_t A_x^{2} A_x^{i-1} u_{i}^{j} + \mu^2 \delta_x^3 u_{i}^{j} + \frac{1}{2} \eta \delta_x (A_x u_{i}^{j})^2 = 0, \]

which has the following discrete multi-symplectic conservation law

\[ \delta_t (A_x d \phi_{i}^{j} \wedge A_x du_{i}^{j-1}) + \delta_x (2d \phi_{i-1}^{j} \wedge dw_{i}^{j} + 2\mu A_x dv_{i}^{j} \wedge A_x du_{i}^{j}) = 0. \]

Applying the Leap-frog rule to time and space derivatives in ODEs (3.8a)-(3.8c), we obtain

\[ \delta_t u_{i+1}^{j} + \delta_t u_{i}^{j-1} + 2\mu^2 \delta_x^3 u_{i}^{j} + \eta \delta_x (u_{i}^{j})^2 = 0, \] (3.25)

which has the following discrete multi-symplectic conservation law

\[ \delta_t (d \phi_{i}^{j} \wedge du_{i}^{j-1}) + \delta_x (2d \phi_{i-1}^{j} \wedge dw_{i}^{j} + 2\mu dv_{i}^{j} \wedge du_{i}^{j}) = 0. \]

Remark 3.2. Scheme (3.25) is the multi-symplectic Euler box scheme for the KdV equation, which has been discussed and verified to show nice numerical stability and the ability to preserve the invariant for long-time integration in [26].

Applying the mid-point rule to space derivatives and Leap-frog rule to time derivatives in ODEs (3.8a)-(3.8c) gives MS VIII

\[ \delta_t A_x^{3} A_x^{i} u_{i}^{j-1} + \mu^2 \delta_x^3 u_{i}^{j} + \frac{1}{2} \eta \delta_x (A_x u_{i}^{j})^2 = 0, \]

which has the following discrete multi-symplectic conservation law

\[ \delta_t (A_x d \phi_{i}^{j} \wedge A_x du_{i}^{j-1}) + \delta_x (2d \phi_{i-1}^{j} \wedge dw_{i}^{j} + 2\mu dv_{i}^{j} \wedge du_{i}^{j}) = 0. \]

3.2. Construction of local energy-conserving algorithms

To illustrate clearly how to derive the discrete local energy conservation law, we give the conservation law in the continuous case first. Multiplying the first line of Eq. (3.1) by \( u_t \) yields

\[ \phi_t u_t = -2w_x u_t + 2wu_t - 2V'(u)u_t, \]

Multiplying the second line of Eq. (3.1) by \( \phi_t \) yields

\[ u_t \phi_t = -2w_x \phi_t. \]
Making the subtraction between the two equations above yields
\[
\varphi_t u_t - u_t \varphi_t = -2\mu v x u_t + 2w u_t - 2V'(u)u_t + 2w x \varphi_t,
\]
\[
0 = \mu v x u_t - w u_t - w x \varphi_t + V_t,
\]
\[
0 = \partial_x(\mu v x u_t) - v v_t - \partial_x(\varphi u_t) + V_t,
\]
\[
0 = \partial_x(\mu v x u_t - w \varphi_t) + \partial_t(V - \frac{1}{2}v_x^2),
\]
i.e.,
\[
\partial_t \left( -\frac{1}{2}v_x^2 + \frac{1}{6}\eta u^3 \right) + \partial_x(-w \varphi_t + \mu v x u_t) = 0,
\] (3.26)
where the term in the first bracket is the energy density and the one in the second bracket is the energy flux. It is worth noting that the Eq. (3.26) is the local energy conservation law for the KdV equation (1.1), which is independent of the boundary conditions. That is to say, it is a more essential property than the global energy conservation law
\[
\int_{x_L}^{x_R} \left( \frac{1}{6}\eta u^3 - \frac{1}{2}v_x^2 \right) dx = \text{Constant},
\] (3.27)
which can be obtained by integrating Eq. (3.26) over the spatial region \([x_L, x_R]\) under suitable boundary conditions, such as periodic or homogeneous boundary conditions. As constructing numerical algorithms for the equation, we hope they can preserve the local energy in any local time-space region. Next, we will present how to construct a series of local energy-conserving algorithms for the KdV equation.

3.2.1. Local energy-conserving scheme I (LECS I)

Discretizing ODE (3.8a) by using the Leap-frog rule, ODEs (3.8b)-(3.8c) using the mid-point rule and the nonlinear term with discrete chain rule in time direction gives
\[
\delta t x u^j_i = a^j_i, \quad \delta x u^j_i = \frac{1}{\mu} v^j_{i+1},
\] (3.28a)
\[
\delta x w^j_i = A_x b^j_i, \quad \delta x \varphi^j_i = A_x w^j_i,
\] (3.28b)
\[
\begin{cases}
\delta_t A_x \varphi^j_i = -2\mu A_x A_t u^j_i + 2A_x A_t w^j_i - 2\delta_t V(A_x u^j_i) \\
\delta_t A_x w^j_i = -2A_x A_t b^j_i.
\end{cases}
\] (3.28c)

Eliminating the auxiliary variables \(a\) and \(b\) yields
\[
\begin{cases}
\delta_t A_x \varphi^j_i = -2\mu A_x A_t u^j_i + 2A_x A_t w^j_i - 2\delta_t V(A_x u^j_i) \\
\delta_t A_x u^j_i = -2\delta_x A_t w^j_i \\
\delta_x w^j_i = \frac{1}{\mu} v^j_{i+1}, \quad \delta_x \varphi^j_i = A_x u^j_i.
\end{cases}
\] (3.29)

Eliminating \(v, w\) and \(\varphi\) yields an equivalent scheme
\[
\delta_t A_x^2 u^j_i + \mu^2 \delta_x^2 A_x A_t u^j_{i-1} + \delta_x \frac{\delta_t V(A_x u^j_i)}{\delta_t A_x u^j_i} = 0.
\]
Theorem 3.5. Scheme (3.29) is a local energy-conserving scheme, which admits the following
local energy conservation law
\[ \delta_t \left( -\frac{1}{2} (A_x v_{i+1}^j)^2 + V(A_x w_{i}^j) \right) + \delta_x (-A_x w_{i}^j \cdot \delta_t \varphi_{i+1}^j + \mu A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j) = 0. \] (3.30)

Proof. Multiplying the first line of Eq. (3.29) by \( \delta_t A_x v_{i}^j \) yields
\[
\delta_t A_x \varphi_{i+1}^j \cdot \delta_t A_x u_{i}^j = -2\mu \delta_x A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j + 2A_x A_t v_{i}^j \cdot \delta_x A_x u_{i}^j - 2\delta_t V(A_x u_{i}^j)
\]
\[
= -2\mu \delta_x A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j + 2A_x A_t v_{i}^j \cdot \delta_x A_x u_{i}^j - 2\delta_t V(A_x u_{i}^j) - 2\delta_t V(A_x u_{i}^j).
\]

Multiplying the second line of Eq. (3.29) by \( \delta_t A_x \varphi_{i+1}^j \) yields
\[
\delta_t A_x w_{i}^j \cdot \delta_t A_x \varphi_{i+1}^j = -2\delta_x A_t w_{i}^j \cdot \delta_t A_x v_{i}^j.
\]

Making the subtraction between the two equations above yields
\[
0 = \mu A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j - \delta_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j + \delta_t V(A_x u_{i}^j) - \delta_x A_t w_{i}^j \cdot \delta_t A_x \varphi_{i+1}^j
\]
\[
= \mu \delta_x A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j - \mu A_x A_t v_{i+1}^j \cdot \delta_t A_x u_{i}^j + \delta_t V(A_x u_{i+1}^j) - \delta_x (A_t w_{i}^j \cdot \delta_t A_x \varphi_{i+1}^j)
\]
\[
= \delta_t \left( -\frac{1}{2} (A_x v_{i+1}^j)^2 + V(A_x u_{i}^j) \right) - \delta_x \left( A_t w_{i}^j \cdot \delta_t A_x \varphi_{i+1}^j - \mu A_x A_t v_{i}^j \cdot \delta_t A_x u_{i}^j \right).
\]

The above ensures the theorem. \( \square \)

3.2.2. Local energy-conserving scheme II (LECS II)

Discretizing ODEs (3.8a)-(3.8b) by using the Leap-frog rule, ODE (3.8c) by using the mid-point rule and the nonlinear term with discrete chain rule in time direction gives
\[
\delta_t v_{i}^j = a_t^j, \quad \delta_x u_{i}^j = \frac{1}{\mu} v_{i+1}^j, \quad \delta_x w_{i}^j = b_t^j, \quad \delta_x \varphi_{i}^j = w_{i+1}^j,
\]
\[
\left\{
\begin{aligned}
\delta_t \varphi_{i}^j &= -2\mu A_x A_t v_{i}^j + 2A_t w_{i}^j - \frac{2\delta_t V(A_x u_{i}^j)}{\delta_x A_t w_{i}^j}, \\
\delta_t u_{i}^j &= -2A_t b_t^j.
\end{aligned}
\right. \quad \text{(3.31c)}
\]

Eliminating the auxiliary variables \( a \) and \( b \) yields
\[
\left\{
\begin{aligned}
\delta_t v_{i}^j &= -2\mu A_x A_t v_{i}^j + 2A_t w_{i}^j - \frac{2\delta_t V(A_x u_{i}^j)}{\delta_x A_t w_{i}^j}, \\
\delta_t u_{i}^j &= -2A_t A_x v_{i}^j, \\
\delta_x u_{i}^j &= \frac{1}{\mu} v_{i+1}^j, \quad \delta_x \varphi_{i}^j = u_{i+1}^j.
\end{aligned}
\right. \quad \text{(3.32)}
\]

Eliminating \( v, w \) and \( \varphi \) yields an equivalent scheme
\[
\delta_t A_x u_{i}^j + \mu^2 \delta_x^3 A_t u_{i-1}^j + \delta_x \frac{\delta_t V(A_x u_{i}^j)}{\delta_x A_t u_{i}^j} = 0.
\]

Theorem 3.6. Scheme (3.32) is a local energy-conserving scheme, which admits the following
local energy conservation law
\[
\delta_t \left( -\frac{1}{2} (v_{i+1}^j)^2 + V_t^j \right) + \delta_x (-A_t u_{i}^j \cdot \delta_t \varphi_{i-1}^j + \mu A_t v_{i}^j \cdot \delta_t A_x u_{i}^j) = 0. \quad \text{(3.33)}
\]
Eliminating the auxiliary variables $v^j_i$ on both sides of the first line of Eq. (3.32) yields
\[
\delta_t \varphi^j_i \cdot \delta_t u^j_i = -2 \mu \delta_x A_x v^j_i \cdot \delta_t u^j_i + 2 A_x u^j_i \cdot \delta_t v^j_i - 2 \delta_t (V^j_i) \\
= -2 \mu \delta_x A_x v^j_i \cdot \delta_t u^j_i + 2 A_x u^j_i \cdot \delta_x \varphi^j_{i-1} - 2 \delta_t (V^j_i).
\]
Taking the product with $\delta_t \varphi^j_i$ on both sides of the second line of Eq. (3.32) yields
\[
\delta_t u^j_i \cdot \delta_t \varphi^j_i = -2 \delta_x A_x u^j_i \cdot \delta_t \varphi^j_i.
\]
Making the subtraction between the two equations above yields
\[
0 = \mu \delta_x A_x v^j_i \cdot \delta_t u^j_i - A_x u^j_i \cdot \delta_x \delta_t \varphi^j_{i-1} + \delta_t (V^j_i) - \delta_x A_x u^j_i \cdot \delta_t \varphi^j_i \\
= \mu \delta_x (A_x v^j_i \cdot \delta_t u^j_i) - \mu A_x u^j_i \cdot \delta_x \delta_t u^j_i + \delta_t (V^j_i) - \delta_x (A_x u^j_i \cdot \delta_t \varphi^j_{i-1}) \\
= \delta_t \left(- \frac{1}{2} (V^j_{i+1})^2 + V^j_i\right) - \delta_x (A_x u^j_i \cdot \delta_t \varphi^j_{i-1} - \mu A_x u^j_i \cdot \delta_t u^j_i). \\
\]
The above ensures the theorem. \qed

### 3.2.3. Local energy-conserving scheme III (LECS III)

Discretizing ODEs (3.8a), (3.8c) by using the mid-point rule, ODE (3.8b) by using the Leap-frog rule and the nonlinear term with discrete chain rule in time direction gives

\[
\begin{align*}
\delta_x v^j_i &= A_x a^j_i, \quad \delta_x u^j_i = \frac{1}{\mu} A_x v^j_i, \quad \delta_x \varphi^j_i = u^j_{i+1}, \\
\delta_x w^j_i &= b^j_i, \quad \delta_x v^j_i = u^j_{i+1}, \\
\delta_t A_x \varphi^j_i &= -2 \mu A_x A_x v^j_i + 2 A_x A_x u^j_i - 2 \frac{\delta \nu (A_x a^j_i)}{\delta A_x a^j_i}, \\
\delta_t A_x u^j_i &= -2 A_x A_x b^j_i. \\
\end{align*}
\]

Eliminating the auxiliary variables $a$ and $b$ yields

\[
\begin{align*}
\delta_t A_x \varphi^j_i &= -2 \mu A_x A_x v^j_i + 2 A_x A_x u^j_i - 2 \frac{\delta \nu (A_x a^j_i)}{\delta A_x a^j_i}, \\
\delta_t A_x u^j_i &= -2 A_x A_x b^j_i, \\
\delta_x w^j_i &= \frac{1}{\mu} A_x v^j_i, \quad \delta_x \varphi^j_i = u^j_{i+1}.
\end{align*}
\]

Eliminating $v$, $w$ and $\varphi$ yields an equivalent scheme

\[
\delta_t A_x^3 u^j_i + \mu^2 \delta_x^2 A_x u^j_i + \delta_x A_x \frac{\delta \nu (A_x u^j_i)}{\delta A_x u^j_i} = 0.
\]

**Theorem 3.7.** Scheme (3.35) is a local energy-conserving scheme, which admits the following local energy conservation law

\[
\delta_t \left(- \frac{1}{2} (A_x v^j_i)^2 + V(A_x u^j_i)\right) + \delta_x (-A_x A_x u^j_i \cdot \delta_t A_x \varphi^j_{i-1} + \mu A_x v^j_i \cdot \delta_t u^j_i) = 0.
\]

**Proof.** Multiplying the first line of Eq. (3.35) by $\delta_t A_x u^j_i$ yields

\[
\delta_t A_x \varphi^j_i \cdot \delta_t A_x u^j_i = -2 \mu \delta_x A_x v^j_i \cdot \delta_t A_x u^j_i + 2 A_x A_x u^j_i \cdot \delta_t A_x \varphi^j_{i-1} - 2 \delta_t V(A_x u^j_i) \\
= -2 \mu \delta_x A_x v^j_i \cdot \delta_t A_x u^j_i + 2 A_x A_x u^j_i \cdot \delta_x \delta_t A_x \varphi^j_{i-1} - 2 \delta_t V(A_x u^j_i).
\]
Multiplying the second line of Eq. (3.35) by $\delta_t A_x \varphi_i^j$ yields
\[ \delta_t A_x u_i^j \cdot \delta_t A_x \varphi_i^j = -2\delta_x A_x A_i w_i^j \cdot \delta_t A_x \varphi_i^j. \]

Making the subtraction between the two equations above yields
\[ 0 = \mu \delta_x A_i v_i^j \cdot \delta_t A_x u_i^j - A_x A_i w_i^j \cdot \delta_x \delta_t A_x \varphi_i^j \delta_x V(A_x u_i^j) - \delta_x A_x A_i w_i^j \cdot \delta_t A_x \varphi_i^j \]
\[ = \mu \delta_x (A_i v_i^j \cdot \delta_t u_i^j) - \mu A_x A_i v_i^j \cdot \delta_x \delta_t u_i^j + \delta_t V(A_x u_i^j) - \delta_x (A_x A_i w_i^j \cdot \delta_t A_x \varphi_i^j - 1) \]
\[ = \delta_t \left( -\frac{1}{2}(A_i v_i^j)^2 + V(A_x u_i^j) - \delta_x \left( A_x A_i w_i^j \cdot \delta_t A_x \varphi_i^j - 1 - \mu A_t v_i^j \cdot \delta_t u_i^j \right) \right). \]

The above ensures the theorem. \[ \square \]

### 3.2.4. Local energy-conserving scheme IV (LECS IV)

Discretizing ODEs (3.8a)-(3.8c) by using the mid-point rule and the nonlinear term with discrete chain rule in time direction gives
\[
\begin{align*}
\delta_x v_i^j &= A_x u_i^j, \quad \delta_x w_i^j = \frac{1}{\mu} A_x v_i^j, \quad \text{(3.37a)} \\
\delta_x w_i^j &= A_x b_i^j, \quad \delta_x \varphi_i^j = A_x u_i^j, \quad \text{(3.37b)} \\
\{ \delta_t A_x \varphi_i^j &= -2\mu A_x A_i v_i^j + 2A_x A_i w_i^j - 2A_x \frac{\delta V(A_x u_i^j)}{\delta_x A_x u_i^j}, \quad \text{(3.37c)} \\
\delta_t A_x u_i^j &= -2A_x A_i b_i^j. \end{align*}
\]

Eliminating the auxiliary variables $a$ and $b$ yields
\[
\begin{align*}
\{ \delta_t A_x \varphi_i^j &= -2\mu \delta_x A_i v_i^j + 2A_x A_i w_i^j - 2A_x \frac{\delta V(A_x u_i^j)}{\delta_x A_x u_i^j}, \\
\delta_t A_x u_i^j &= -2\delta_x A_i w_i^j, \\
\delta_x u_i^j &= \frac{1}{\mu} A_x v_i^j, \quad \delta_x \varphi_i^j = A_x u_i^j. \end{align*}
\]

Eliminating $v$, $w$, and $\varphi$ yields an equivalent scheme
\[
\delta_t A_x^2 u_i^j + \mu^2 \delta^2 A_i u_i^j + \delta_x A_x \frac{\delta V(A_x u_i^j)}{\delta_t A_x u_i^j} = 0.
\]

**Theorem 3.8.** Scheme (3.38) is a local energy-conserving scheme, which admits the following local energy conservation law
\[
\delta_t \left( -\frac{1}{2}(A_x v_i^j)^2 + V(A_x u_i^j) \right) + \delta_x (-A_t w_i^j \cdot \delta_t \varphi_i^j + \mu A_t v_i^j \cdot \delta_t u_i^j) = 0. \quad \text{(3.39)}
\]

**Proof.** Taking the product with $\delta_t A_x u_i^j$ on both sides of the first line of Eq. (3.38) gives
\[
\begin{align*}
\delta_t A_x \varphi_i^j \cdot \delta_t A_x u_i^j &= -2\mu \delta_x A_i v_i^j \cdot \delta_t A_x u_i^j + 2A_x A_t w_i^j \cdot \delta_t A_x u_i^j - 2\delta_t V(A_x u_i^j) \\
&= -2\mu \delta_x A_t v_i^j \cdot \delta_t A_x w_i^j + 2A_x A_t w_i^j \cdot \delta_t \delta_x \varphi_i^j - 2\delta_t V(A_x u_i^j).
\end{align*}
\]

Taking the product with $\delta_t A_x \varphi_i^j$ on both sides of the second line of Eq. (3.38) gives
\[
\begin{align*}
\delta_t A_x u_i^j \cdot \delta_t A_x \varphi_i^j &= -2\delta_x A_t w_i^j \cdot \delta_t A_x \varphi_i^j.
\end{align*}
\]
Making the subtraction between the two equations above yields

\[
0 = \mu t_x A_i v_i^j \cdot \delta_t A_i u_i^j - A_x A_i v_i^j \cdot \delta_t \phi_i^j + \delta_t V(A_x u_i^j) - \delta_x A_i u_i^j \cdot \delta_t A_x \phi_i^j
\]

\[
= \mu \delta_x (A_x v_i^j \cdot \delta_t \phi_i^j) - \mu A_x A_i v_i^j \cdot \delta_t \phi_i^j + \delta_t V(A_x u_i^j) - \delta_x (A_t u_i^j \cdot \delta_t \phi_i^j)
\]

\[
= \delta_t \left( -\frac{1}{2} (A_x v_i^j)^2 + V(A_x u_i^j) \right) - \delta_x (A_t u_i^j \cdot \delta_t \phi_i^j - \mu A_t v_i^j \cdot \delta_t u_i^j).
\]

The above ensures the theorem. \[\square\]

### 3.2.5. Local energy-conserving scheme V-VIII (LECS V-VIII)

In this subsection, we will present the other four local energy-conserving schemes and we also do not repeat the detailed process here due to the space limitations. Discretizing ODE (3.8a) by using the mid-point rule, ODEs (3.8b)-(3.8c) by using the Leap-frog rule and the nonlinear term with discrete chain rule in time direction, we have **LECS V**

\[
\delta_t A_x^2 A_t u_{i+1}^j + \delta_t A_x^2 A_t u_i^j - 2 \mu^2 \delta_x^2 u_i^j + 2 \delta_x A_x \frac{\delta_x A_t V(A_x u_{i+1}^j)}{\delta_t A_x A_t u_{i+1}^j} = 0,
\]

which admits the following local energy conservation law

\[
\delta_t \left( -\frac{1}{2} (A_x v_i^j)^2 + A_t V(A_x u_i^j) \right) + \delta_x (A_t v_i^j \cdot \delta_t \phi_i^j - \mu A_t v_i^j \cdot \delta_t A_x \phi_i^j) = 0.
\]

Discretizing ODEs (3.8a) and (3.8c) by using the Leap-frog rule, ODE (3.8b) by using the mid-point rule and the nonlinear term with discrete chain rule in time direction, we have **LECS VI**

\[
\delta_t A_x^2 A_t u_{i+1}^j + \mu^2 \delta_x^2 A_t u_i^j + 2 \delta_x A_x \frac{\delta_t A_x V(A_x u_{i+1}^j)}{\delta_t A_x A_t u_{i+1}^j} = 0,
\]

which possesses the following local energy conservation law

\[
\delta_t \left( -\frac{1}{2} (A_x v_{i+1}^j)^2 + A_t V(A_x u_{i+1}^j) \right) + \delta_x (A_t v_{i+1}^j \cdot \delta_t \phi_{i+1}^j - \mu A_t v_i^j \cdot \delta_t A_x \phi_{i+1}^j) = 0.
\]

Discretizing ODEs (3.8a)-(3.8c) by using the Leap-frog rule and the nonlinear term with discrete chain rule in time direction gives **LECS VII**

\[
\delta_t A_x^2 A_t u_{i+1}^j + \delta_t A_t u_i^j + 2 \mu^2 \delta_x^2 u_i^j + 2 \delta_x A_x \frac{\delta_t A_x V(A_x u_{i+1}^j)}{\delta_t A_x A_t u_{i+1}^j} = 0,
\]

which admits the following local energy conservation law

\[
\delta_t \left( -\frac{1}{2} v_{i+1}^j \cdot v_i^j + A_t V_i^j \right) + \delta_x (A_t v_i^j \cdot \delta_t \phi_i^j - \mu v_i^j \cdot \delta_t A_x \phi_i^j) = 0.
\]

Discretizing ODEs (3.8a)-(3.8b) by using the mid-point rule, ODE (3.8c) by using the Leap-frog rule and the nonlinear term with discrete chain rule in time direction gives **LECS VIII**

\[
\delta_t A_x^3 A_t u_{i+1}^j + \mu^2 \delta_x^2 u_i^j + 2 \delta_x A_x \frac{\delta_t A_x V(A_x u_{i+1}^j)}{\delta_t A_x A_t u_{i+1}^j} = 0,
\]

which possesses the following local energy conservation law

\[
\delta_t \left( -\frac{1}{2} (A_x v_{i+1}^j)^2 + A_t V(A_x u_{i+1}^j) \right) + \delta_x (A_t v_{i+1}^j \cdot \delta_t \phi_{i+1}^j - \mu v_{i+1}^j \cdot \delta_t A_x \phi_{i+1}^j) = 0.
\]
3.3. Construction of local momentum-conserving algorithms

The local momentum conservation law for KdV equation (1.1) is

$$\partial_t \left( -\frac{1}{2} u^2 \right) + \partial_x \left( \frac{1}{2} v^2 - wu + V + \frac{1}{2} \varphi u \right) = 0,$$

(3.40)

where the term in the first bracket is the momentum density and the one in the second bracket is the momentum flux. With periodic or homogeneous boundary conditions, we have the global momentum conservation law

$$\int_{x_L}^{x_R} \left( -\frac{1}{2} u^2 \right) dx = \text{Constant}.\quad (3.41)$$

Momentum conservation law is also an important invariant in physics. To our knowledge, there is few momentum-conserving methods for KdV equation in literatures. Therefore, to construct algorithms, which possess the momentum conservation law, is important and interesting. Next, we will construct eight local momentum-conserving algorithms for the KdV equation.

3.3.1. Local momentum-conserving scheme I (LMCS I)

Discretizing ODEs (3.8b)-(3.8c) by using the mid-point rule, ODE (3.8a) by using the Leap-frog rule and the nonlinear term with discrete chain rule in spacial direction gives

$$\begin{align*}
\delta_x v_i^j &= u_i^j, \quad \delta_x u_i^j = \frac{1}{\mu} v_{i+1}^j, \quad (3.42a) \\
\delta_x w_i^j &= A_x b_i^j, \quad \delta_x \varphi_i^j = A_x u_i^j, \quad (3.42b) \\
\begin{cases}
\delta_t A_x \varphi_i^j = -2\mu A_x A_t u_i^j + 2A_x A_t w_i^j - \frac{2}{\mu} \frac{\delta_x V(A_t u_i^j)}{\delta_x A_t u_i^j}, \\
\delta_t A_x u_i^j = -2A_x A_t b_i^j.
\end{cases} \quad (3.42c)
\end{align*}$$

Eliminating the auxiliary variables $a$ and $b$ yields

$$\begin{align*}
\delta_t A_x \varphi_i^j &= -2\mu \delta_x A_x A_t u_i^j + 2A_x A_t w_i^j - \frac{2}{\mu} \frac{\delta_x V(A_t u_i^j)}{\delta_x A_t u_i^j}, \\
\delta_t A_x u_i^j &= -2\delta_x A_t u_i^j, \\
\delta_x w_i^j &= \frac{1}{\mu} v_{i+1}^j, \quad \delta_x \varphi_i^j = A_x u_i^j. \quad (3.43)
\end{align*}$$

Eliminating $v$, $w$ and $\varphi$ yields an equivalent scheme

$$\delta_t A_x^2 u_i^j + \mu^2 \delta_x^2 A_x A_t u_i^j - \mu \frac{\delta_x V(A_t u_i^j)}{\delta_x A_t u_i^j} = 0.\quad (3.44)$$

**Theorem 3.9.** Scheme (3.43) is a local momentum-conserving scheme, which admits the following local momentum conservation law

$$\delta_t \left( -\frac{1}{2} (A_x u_i^j)^2 \right) + \delta_x \left( \frac{1}{2} A_t v_i^j \cdot A_t v_{i+1}^j - A_t u_i^j \cdot A_t u_{i+1}^j + V(A_t u_i^j) + \frac{1}{2} \delta_x \varphi_i^j \cdot A_t u_i^j \right) = 0.$$

**Proof.** Multiplying the first line of Eq. (3.43) by $\delta_x A_t u_i^j$ yields

$$\delta_t A_x \varphi_i^j \cdot \delta_x A_t u_i^j = -2\mu \delta_x A_x A_t u_i^j \cdot \delta_x A_t u_i^j + 2A_x A_t w_i^j \cdot \delta_x A_t u_i^j - 2\delta_x V(A_t u_i^j).$$
Multiplying the second line of Eq. (3.43) by δxA_i δϕ_i yields
\[ \delta_t A_x u^i_t \cdot \delta_x A_t \varphi^i_t = -2\delta_x A_t w^i_t \cdot \delta_x A_t \varphi^i_t. \]

Making the subtraction between the two equations above yields
\[
\begin{align*}
\text{LHS} &= \delta_t A_x \varphi^i_t \cdot \delta_x A_t u^i_t - \delta_t A_x u^i_t \cdot \delta_x A_t \varphi^i_t \\
&= \delta_t A_x \varphi^i_t \cdot \delta_x A_t u^i_t + \delta_x \delta_t \varphi^i_t \cdot A_x A_t u^i_t - \delta_x \delta_t \varphi^i_t \cdot A_x A_t u^i_t - \delta_t A_x u^i_t \cdot \delta_x A_t \varphi^i_t \\
&= \delta_x (\delta_t \varphi^i_t \cdot A_t u^i_t) + \delta_t (-\frac{1}{\mu}(A_x u^i_t)^2), \\
\text{RHS} &= -2\mu \delta_x A_x A_t v^{i+1}_j \cdot \delta_x A_t u^i_t + 2A_x A_t w^i_t \cdot \delta_x A_t u^i_t + 2\delta_x A_t w^i_t \cdot \delta_x A_t \varphi^i_t - 2\delta_x V(A_t u^i_t) \\
&= -2\delta_x A_x A_t v^{i+1}_j \cdot \delta_x A_t u^i_t + 2A_x A_t w^i_t \cdot \delta_x A_t u^i_t + 2\delta_x A_t w^i_t \cdot A_x A_t u^i_t - 2\delta_x V(A_t u^i_t) \\
&= \delta_x (-A_t u^i_t \cdot A_t v^{i+1}_t + 2A_x A_t w^i_t \cdot A_t u^i_t - 2V(A_t u^i_t)),
\end{align*}
\]
which implies the local momentum conservation law (3.44). □

3.3.2. Local momentum-conserving scheme II (LMCS II)

Discretizing ODEs (3.8a)-(3.8b) by using the Leap-frog rule, ODE (3.8c) by using the mid-point rule and the nonlinear term with discrete chain rule in spacial direction gives
\[
\begin{align*}
\delta_x v^i_t &= u^i_t, \quad \delta_x u^i_t = \frac{1}{\mu} v^{i+1}_j, \\
\delta_x w^i_t &= b^i_t, \quad \delta_x \varphi^i_t = v^{i+1}_j, \\
\delta_t A_x \varphi^i_t &= -2\mu A_x A_t v^{i+1}_j + 2A_x A_t w^i_t - 2\frac{\delta_x V(A_t u^i_t)}{\delta_x A_t u^i_t}, \\
\delta_t A_x A_t u^i_t &= -2A_x A_t b^i_t. \\
\end{align*}
\]

Eliminating the auxiliary variables a and b yields
\[
\begin{align*}
\delta_t A_x \varphi^i_t &= -2\mu \delta_x A_x A_t v^{i+1}_j + 2A_x A_t w^i_t - 2\frac{\delta_x V(A_t u^i_t)}{\delta_x A_t u^i_t}, \\
\delta_t A_x A_t u^i_t &= -2\delta_x A_x A_t v^{i+1}_j, \\
\delta_x v^i_t &= \frac{1}{\mu} v^{i+1}_j, \quad \delta_x \varphi^i_t = v^{i+1}_j.
\end{align*}
\]

Eliminating v, w and ϕ yields an equivalent scheme
\[ \delta_t A_x^2 u^i_t + \mu^2 \delta_x^2 A_x A_t u^i_{t-1} + \delta_x \frac{\delta_x V(A_t u^i_t)}{\delta_x A_t u^i_t} = 0. \]

**Theorem 3.10.** Scheme (3.46) is a local momentum-conserving scheme, which admits the following local momentum conservation law
\[
\delta_t \left( -\frac{1}{2} v^{i+1}_j \cdot A_x u^i_t \right) + \delta_x \left( \frac{1}{2} A_t v^{i+1}_j \cdot A_t u^i_t - A_x A_t w^i_t \cdot A_t u^i_t \right) + V(A_t u^i_t) + \frac{1}{2} \delta_t \varphi^i_t \cdot A_t u^i_t = 0.
\]

**Proof.** Taking the product with δxA_t u^i_t on both sides of the first line of Eq. (3.46) yields
\[ \delta_t A_x \varphi^i_t \cdot \delta_x A_t u^i_t = -2\mu \delta_x A_x A_t v^{i+1}_j \cdot \delta_x A_t u^i_t + 2A_x A_t w^i_t \cdot \delta_x A_t u^i_t - 2\mu V(A_t u^i_t). \]
Taking the product with $\delta_x A_i \varphi^j_i$ on both sides of the second line of Eq. (3.46) yields

$$\delta_t A_x u^j_i \cdot \delta_x A_i \varphi^j_i = -2 \delta_x A_x A_i u^j_i \cdot \delta_x A_i \varphi^j_i.$$ 

Making the subtraction between the two equations above yields

$$\begin{align*}
LHS &= \delta_t A_x \varphi^j_i \cdot \delta_x A_i u^j_i - \delta_t A_x u^j_i \cdot \delta_x A_i \varphi^j_i \\
&= \delta_t A_x \varphi^j_i \cdot \delta_x A_i u^j_i + \delta_x \delta_t \varphi^j_i \cdot A_x A_i u^j_i - \delta_x \delta_t \varphi^j_i \cdot A_x A_i u^j_i - \delta_t A_x u^j_i \cdot \delta_x A_i \varphi^j_i \\
&= \delta_x (\delta_t \varphi^j_i \cdot A_t u^j_i) + \delta_t (-u^j_{i+1} \cdot A_x u^j_i),
\end{align*}$$

$$\begin{align*}
RHS &= -2\delta_x A_x A_i v^j_i \cdot \delta_x A_i u^j_i + 2 A_x A_i \varphi^j_i \cdot \delta_x A_i u^j_i + 2 \delta_x A_x A_i \varphi^j_i \cdot \delta_x A_i u^j_i - 2 \delta_x V(A_i u^j_i) \\
&= -2 \delta_x A_x A_i v^j_i \cdot A_t v^j_{i+1} + 2 A_x A_i \varphi^j_i \cdot \delta_x A_d u^j_i + 2 \delta_x A_x A_i \varphi^j_i \cdot A_t v^j_{i+1} - 2 \delta_x V(A_t u^j_i) \\
&= \delta_x (-A_t v^j_i \cdot A_t v^j_{i+1} + 2 A_x A_i \varphi^j_i \cdot A_t v^j_{i+1} - 2 V(A_t u^j_i))
\end{align*}$$

which implies the local momentum conservation law (3.47). \hfill \Box

### 3.3.3. Local momentum-conserving scheme III (LMCS III)

Discretizing ODEs (3.8a) and (3.8c) by using the mid-point rule, ODE (3.8b) by using the Leap-frog rule and the nonlinear term with discrete chain rule in spacial direction gives

$$\begin{align*}
\delta_x v^j_i &= A_x a^j_i, & \delta_x u^j_i &= \frac{1}{\mu} A_x v^j_i, \\
\delta_x w^j_i &= b^j_i, & \delta_x \varphi^j_i &= u^j_{i+1}, \\
\delta_t A_x \varphi^j_i &= -2\mu \delta_x A_t a^j_i + 2 A_x A_t u^j_i - 2 \mu V(A_t u^j_i), \\
\delta_t A_x u^j_i &= -2 A_x A_t b^j_i. \\
\delta_x u^j_i &= \frac{1}{\mu} A_x v^j_i, & \delta_x \varphi^j_i &= u^j_{i+1}.
\end{align*}$$

(3.48a, 3.48b, 3.48c)

Eliminating the auxiliary variables $a$ and $b$ yields

$$\begin{align*}
\delta_t A_x \varphi^j_i &= -2\mu \delta_x A_t a^j_i + 2 A_x A_t u^j_i - 2 \mu V(A_t u^j_i), \\
\delta_t A_x u^j_i &= -2 \delta_x A_x A_t \varphi^j_i, \\
\delta_x u^j_i &= \frac{1}{\mu} A_x v^j_i, & \delta_x \varphi^j_i &= u^j_{i+1}.
\end{align*}$$

(3.49)

Eliminating $v, w$ and $\varphi$ yields an equivalent scheme

$$\delta_t A_x^3 u_i^j + \mu^2 \delta_x A_t A_i u^j_i + \delta_x A_i \frac{\delta_x V(A_t u^j_i)}{\delta_x A_t u^j_i} = 0.$$

**Theorem 3.11.** Scheme (3.49) is a local momentum-conserving scheme, which possesses the following local momentum conservation law

$$\delta_t \left( \left( \frac{1}{2} A_t a^j_i \cdot A_x u^j_i \right) + \mu \left( \frac{1}{2} (A_t v^j_i)^2 - A_x A_t w^j_i \cdot A_t u^j_i + V(A_t u^j_i) + \frac{1}{2} \delta_t \varphi^j_i \cdot A_t u^j_i \right) \right) = 0. $$

(3.50)

**Proof.** Multiplying the first line of Eq. (3.49) by $\delta_x A_t u^j_i$ yields

$$\delta_t A_x \varphi^j_i \cdot \delta_x A_t u^j_i = -2 \mu \delta_x A_t a^j_i \cdot \delta_x A_t u^j_i + 2 A_x A_t u^j_i \cdot \delta_x A_t u^j_i - 2 \delta_x V(A_t u^j_i).$$
Multiplying the second line of Eq. (3.49) by $\delta_x A_t \varphi^1_i$ yields

$$\delta_t A_x u_i^1 \cdot \delta_x A_t \varphi^1_i = -2\delta_x A_x A_t u_i^1 \cdot \delta_x A_t \varphi^1_i.$$  

Making the subtraction between the two equations above yields

$$\text{LHS} = \delta_t A_x \varphi^1_i \cdot \delta_x A_t u_i^1 - \delta_t A_x u_i^1 \cdot \delta_x A_t \varphi^1_i$$

$$= \delta_t A_x \varphi^1_i \cdot \delta_x A_t u_i^1 + \delta_x \delta_t \varphi^1_i \cdot A_x A_t u_i^1 - \delta_x \delta_t \varphi^1_i \cdot A_x A_t u_i^1 - \delta_t A_x u_i^1 \cdot \delta_x A_t \varphi^1_i$$

$$= \delta_x (\delta_t \varphi^1_i \cdot A_t u_i^1) + \delta_x (-u_{i+1}^1 \cdot A_x u_i^1),$$

$$\text{RHS} = -2\mu \delta_x A_t u_i^1 \cdot \delta_x A_t u_i^1 + 2A_x A_t u_i^1 \cdot \delta_x A_t u_i^1 + 2\delta_x A_x A_t u_i^1 \cdot \delta_x A_t \varphi^1_i - 2\delta_x V(A_t u_i^1)$$

$$= -2\delta_x A_t \varphi^1_i \cdot A_x A_t u_i^1 + 2A_x A_t u_i^1 \cdot \delta_x A_t u_i^1 + 2\delta_x A_x A_t u_i^1 \cdot A_t u_{i+1}^1 - 2\delta_x V(A_t u_i^1)$$

$$= \delta_x (-\delta_t \varphi^1_i)^2 + 2A_x A_t u_i^1 \cdot A_t u_i^1 - 2V(A_t u_i^1),$$

which implies the local momentum conservation law (3.50).

3.3.4. Local momentum-conserving scheme IV (LMCS IV)

Discretizing ODEs (3.8a)-(3.8c) by using the mid-point rule and the nonlinear term with discrete chain rule in spacial direction gives

$$\delta_x u_i^1 = A_x a_i^1, \quad \delta_x u_i^1 = \frac{1}{\mu} A_x u_i^1, \quad \delta_x u_i^1 = \frac{1}{\mu} A_x u_i^1,$$

$$\delta_x u_i^1 = A_x b_i^1, \quad \delta_x u_i^1 = A_x u_i^1,$$

$$\delta_x u_i^1 = -2\mu A_x A_t a_i^1 + 2A_x A_t u_i^1 - \frac{2\delta_x V(A_t u_i^1)}{\delta_x A_t u_i^1},$$

$$\delta_t A_x \varphi^1_i = -2\mu \delta_x A_t u_i^1 + 2A_x A_t u_i^1 - \frac{2\delta_x V(A_t u_i^1)}{\delta_x A_t u_i^1},$$

$$\delta_x u_i^1 = \frac{1}{\mu} A_x u_i^1, \quad \delta_x \varphi^1_i = A_x u_i^1.$$  

Eliminating the auxiliary variables $a$ and $b$ yields

$$\delta_x A_x \varphi^1_i = -2\mu \delta_x A_t u_i^1 + 2A_x A_t u_i^1 - \frac{2\delta_x V(A_t u_i^1)}{\delta_x A_t u_i^1},$$

$$\delta_t A_x u_i^1 = -2\delta_x A_t u_i^1,$$

$$\delta_x u_i^1 = \frac{1}{\mu} A_x u_i^1, \quad \delta_x \varphi^1_i = A_x u_i^1.$$  

Eliminating $v$, $w$ and $\varphi$ yields an equivalent scheme

$$\delta_t A_x^2 u_i^1 + \mu^2 \delta_x A_x u_i^1 + \delta_x A_x \frac{\delta_x V(A_t u_i^1)}{\delta_x A_t u_i^1} = 0.$$  

Theorem 3.12. Scheme (3.52) is a local momentum-conserving scheme, which possesses the following local momentum conservation law

$$\delta_t \left( -\frac{1}{2} A_x u_i^1 \right)^2 + \delta_x \left( \frac{1}{2} A_t u_i^1 \right)^2 - A_t w_i^1 \cdot A_t u_i^1 + V(A_t u_i^1) + \left( \frac{1}{2} \delta_t \varphi^1_i \cdot A_t u_i^1 \right) = 0.$$  

Proof. Taking the product with $\delta_x A_t u_i^1$ on both sides of the first line of Eq. (3.52) gives

$$\delta_t A_x \varphi^1_i \cdot \delta_x A_t u_i^1 = -2\mu \delta_x A_t u_i^1 \cdot \delta_x A_t u_i^1 + 2A_x A_t u_i^1 \cdot \delta_x A_t u_i^1 - 2\delta_x V(A_t u_i^1).$$
Taking the product with $\delta_x A_i \phi_i^j$ on both sides of the second line of Eq. (3.52) gives

$$\delta_t A_x u_i^j \cdot \delta_x A_i \phi_i^j = -2 \delta_x A_t u_i^j \cdot \delta_x A_i \phi_i^j.$$  

Making the subtraction between the two equations above yields

LHS = $\delta_t A_x \phi_i^j \cdot \delta_x A_t u_i^j - \delta_t A_x u_i^j \cdot \delta_x A_i \phi_i^j$

$= \delta_t A_x \phi_i^j \cdot \delta_x A_t u_i^j + \delta_t \delta_x A_t \phi_i^j \cdot A_x \phi_i^j - \delta_x \delta_t \phi_i^j \cdot A_x \phi_i^j - \delta_t A_x u_i^j \cdot \delta_x A_i \phi_i^j$

$= \delta_x (\delta_t \phi_i^j \cdot A_x u_i^j) + \delta_t \left(-\left(A_t u_i^j \right)^2\right),$

RHS = $-2 \mu \delta_x A_t \phi_i^j \cdot \delta_x A_t u_i^j + 2 A_x A_t \phi_i^j \cdot \delta_x A_t u_i^j + 2 \delta_x A_t \phi_i^j \cdot \delta_x A_t u_i^j - 2 \delta_x V(A_t u_i^j)$

$= -2 \delta_x A_t \phi_i^j \cdot A_x u_i^j + 2 A_x A_t u_i^j \cdot \delta_x A_t u_i^j + 2 \delta_x A_t u_i^j \cdot A_x \phi_i^j - 2 \delta_x V(A_t u_i^j)$

$= \delta_x \left(-\left(A_t u_i^j \right)^2 + 2 A_t \phi_i^j \cdot A_x u_i^j - 2 V(A_t u_i^j)\right),$  

which implies the local momentum conservation law (3.53). \hfill \Box

### 3.3.5. Local momentum-conserving scheme V-VIII (LMCS V-VIII)

Similarly, here we also omit the specific derivation process of the other four local momentum-conserving algorithms. Discretizing ODEs (3.8b)-(3.8c) by using the Leap-frog rule, ODE (3.8a) by using the mid-point rule and the nonlinear term with discrete chain rule in spacial direction, we have **LMCS V**

$$\delta_t A_x^j u_{i+1} + \delta_t A_x^j u_{i-1}^j + 2 \mu \delta_x^3 A_x \phi_i^j \cdot \delta_x \delta_x V(u_i^j) = 0,$$

which admits the following local momentum conservation law

$$\delta_t \left(-\frac{1}{2} u_{i+1}^j \cdot A_x u_i^j - 1\right) + \delta_x \left(\frac{1}{2} \left(u_i^j \right)^2 + A_x u_i^j \cdot u_i^j + V(u_i^j) + \frac{1}{2} \delta_t \phi_i^j \cdot u_i^j \right) = 0.$$

Discretizing ODEs (3.8a) and (3.8c) by using the Leap-frog rule, ODE (3.8b) by using the mid-point rule and the nonlinear term with discrete chain rule in spacial direction, we obtain **LMCS VI**

$$\delta_t A_x \phi_i^j \cdot A_x u_i^j + \mu \delta_x^3 A_x \phi_i^j \cdot \delta_x \delta_x V(u_i^j) = 0,$$

which possesses the following local momentum conservation law

$$\delta_t \left(-\frac{1}{2} A_x \phi_i^j \cdot A_x u_i^j - 1\right) + \delta_x \left(\frac{1}{2} \left(v_i^j \right)^2 + A_x v_i^j \cdot u_i^j + V(u_i^j) + \frac{1}{2} \delta_t \phi_i^j \cdot u_i^j \right) = 0.$$

Discretizing ODEs (3.8a)-(3.8c) by using the Leap-frog rule and the nonlinear term with discrete chain rule in spacial direction, we have **LMCS VII**

$$\delta_t A_x u_{i+1}^j + \delta_t A_x u_{i-1}^j + 2 \mu \delta_x^3 A_x u_i^j + 2 A_x \phi_i^j \cdot \delta_x \delta_x V(u_i^j) = 0,$$

which admits the following local momentum conservation law

$$\delta_t \left(-\frac{1}{2} u_{i+1}^j \cdot A_x u_i^j - 1\right) + \delta_x \left(\frac{1}{2} \left(u_i^j \right)^2 + A_x v_i^j \cdot u_i^j + V(u_i^j) + \frac{1}{2} \delta_t \phi_i^j \cdot u_i^j \right) = 0.$$
Discretizing ODEs (3.8a)-(3.8b) by using the mid-point rule, ODE (3.8c) by using the Leapfrog rule and the nonlinear term with discrete chain rule in spacial direction, we obtain LMCS VIII

\[ \delta_t A_x^2 u_i^j u_i^{j-1} + \mu^2 \delta_x^3 u_i^j + \delta_x A_x \frac{\delta_x V(u_i^j)}{\delta_x u_i^j} = 0, \]

which possesses the following local momentum conservation law

\[ \delta_t \left( -\frac{1}{2} A_x u_i^{j-1} \cdot A_x u_i^j \right) + \delta_x \left( \frac{1}{2} (u_i^j)^2 - u_i^j \cdot u_i^j + V(u_i^j) + \frac{1}{2} \delta_x \phi_i^j \cdot u_i^j \right) = 0. \]

4. Stability Analysis

To study the stability of the numerical method, that is, the sensitivity of the numerical solution to perturbations in the initial data, the von Neumann stability analysis [17] is used. This method is applicable only for linear problems. To apply this method, we assume that

\[ u_i^j = \lambda^j e^{i l k \Delta x} \quad (4.1) \]

is the test function, where \( l = \sqrt{-1}, \lambda \) is the amplification factor and \( k \) is wave number. The necessary condition for stability of the difference system is \(|\lambda| \leq 1\). To apply von Neumann stability analysis properly, we consider the linearized KdV equation

\[ \frac{\partial u}{\partial t} + \mu^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0. \quad (4.2) \]

• The linearized version of the proposed scheme MS I (or LECS I, LMCS I) applied to Eq. (4.2) is

\[ \delta_t A_x^2 u_i^j + \mu^2 \delta_x^3 A_x u_i^{j-1} = 0. \quad (4.3) \]

By substituting (4.1) into scheme (4.3), we get after some manipulation

\[ (\lambda - 1)(1 + \cos(k\Delta x)) + r(\lambda + 1) \left( \sin(2k\Delta x) - 2 \sin(k\Delta x) \right) l = 0, \quad (4.4) \]

where \( r = \mu^2 \Delta t / \Delta x^3 \). According to some simple calculation, we know that \(|\lambda| = 1\). Thus, the scheme (4.3) is unconditionally linear stable in the linear sense according to von Neumann stability analysis.

• The linearized version of the proposed scheme MS II (or LECS II, LMCS II) applied to Eq. (4.2) is

\[ \delta_t A_x^2 u_i^j + \mu^2 \delta_x^3 A_x u_i^{j-1} = 0. \quad (4.5) \]

By substituting (4.1) into scheme (4.5), we get after some manipulation

\[ (\lambda - 1) \cos\left( \frac{1}{2} k \Delta x \right) + r(\lambda + 1) \left( \sin\left( \frac{3}{2} k \Delta x \right) - 3 \sin\left( \frac{1}{2} k \Delta x \right) \right) l = 0, \quad (4.6) \]

where \( r = \mu^2 \Delta t / \Delta x^3 \). It is obvious that \(|\lambda| = 1\). This means that scheme (4.5) is unconditionally linear stable in the linear sense according to von Neumann stability analysis as well.

In a similar way, we find that the proposed scheme MS III (or LECS III, LMCS III) and MS IV (or LECS IV, LMCS IV) applied to Eq. (4.2) are unconditionally linear stable since the roots admit \(|\lambda| = 1\).
5. Numerical Experiments

In this section, we conduct some numerical examples to exhibit the performance of the proposed local structure-preserving algorithms. The contents include: (i) to simulate the propagation of single soliton and the interaction of two-solitary wave. (ii) to test the energy-conserving and momentum-conserving properties of the proposed methods. (iii) to compare the performance of the proposed structure-preserving algorithms. (iv) to compare the proposed multi-symplectic schemes with the ZK scheme [2] over a long time computation.

We note that the discrete systems for the nonlinear Hamiltonian PDEs are solved by using the simple fixed-point iteration method. The KdV equation (1.1) with periodic boundary conditions has global energy conservation law

$$\mathcal{E} = \int Edx = \text{Constant}, \quad E = -\frac{1}{2} \mu^2 u_x^2 + \frac{1}{6} \eta u^3,$$

and global momentum conservation law

$$\mathcal{M} = \int Mdx = \text{Constant}, \quad M = -\frac{1}{2} u^2.$$

According to the energy conservation laws of different energy-conserving schemes, the discrete global energies of the scheme LECS I, LECS II, LECS III and LECS IV are defined as

$$\mathcal{E}_1^n = h \sum_{i=0}^{N-1} \left( -\frac{1}{2} \mu^2 \left( \frac{u_{i+2}^n - u_i^n}{2h} \right)^2 + \frac{1}{6} \eta \left( \frac{u_i^n + u_{i+1}^n}{2} \right)^3 \right),$$

$$\mathcal{E}_2^n = h \sum_{i=0}^{N-1} \left( -\frac{1}{2} \mu^2 \left( \frac{u_{i+1}^n - u_i^n}{h} \right)^2 + \frac{1}{6} \eta (u_i^n)^3 \right),$$

$$\mathcal{E}_3^n = h \sum_{i=0}^{N-1} \left( -\frac{1}{2} \mu^2 \left( \frac{u_{i+1}^n - u_i^n}{h} \right)^2 + \frac{1}{6} \eta \left( \frac{u_i^n + u_{i+1}^n}{2} \right)^3 \right),$$

$$\mathcal{E}_4^n = h \sum_{i=0}^{N-1} \left( -\frac{1}{2} \mu^2 \left( \frac{u_{i+1}^n - u_i^n}{h} \right)^2 + \frac{1}{6} \eta \left( \frac{u_i^n + u_{i+1}^n}{2} \right)^3 \right).$$

According to the momentum conservation laws of different momentum-conserving schemes, we define the discrete global momentums of the scheme LMCS I, LMCS II, LMCS III and LMCS IV as

$$\mathcal{M}_1^n = -\frac{1}{2} h \sum_{i=0}^{N-1} \left( \frac{u_i^n + u_{i+1}^n}{2} \right)^2, \quad \mathcal{M}_2^n = -\frac{1}{2} h \sum_{i=0}^{N-1} u_{i+1}^n \frac{u_i^n + u_{i+1}^n}{2},$$

$$\mathcal{M}_3^n = -\frac{1}{2} h \sum_{i=0}^{N-1} u_i^n \frac{u_i^n + u_{i+1}^n}{2}, \quad \mathcal{M}_4^n = -\frac{1}{2} h \sum_{i=0}^{N-1} \left( \frac{u_i^n + u_{i+1}^n}{2} \right)^2.$$

Define the relatively errors in discrete global energy and momentum as

$$GE_k = \left| \frac{\mathcal{E}_k^n - \mathcal{E}_k^0}{\mathcal{E}_k^n} \right|, \quad GM_k = \left| \frac{\mathcal{M}_k^n - \mathcal{M}_k^0}{\mathcal{M}_k^0} \right|, \quad k = 1, 2, 3, 4.$$
5.1. Example of a single soliton

We consider the KdV equation (1.1) with initial condition
\[ u(x, 0) = 6sech^2\left(\frac{x}{\sqrt{2}}\right). \]  
(5.1)

The parameters are taken as \( \eta = 1, \mu = 1 \). The computations are done on the space interval \([-10, 10]\) and the boundary is taken to be periodic. We take \( N = 120 \), time step \( \Delta t = 0.002 \) and space step \( \Delta x = \frac{20}{120} \).

Fig. 5.1. The propagation of single soliton simulated by the schemes MS I (left) and MS II (right).

Fig. 5.1 provides the propagation of single solitary over the time interval by using the scheme MS I and MS II. As it can be seen from the figure, the propagation is traveling from left to right as required and the shape of the waves is preserved accurately. Note that the figures of numerical solution by using the schemes MS III and MS IV are similar, so here we do not list them repeated because of the limited space. Therefore, the schemes MS I, MS II, MS III and MS IV can simulate the propagation process of single soliton very well.

Moreover, we compare these new schemes on the performance of preserving the invariants (energy and momentum) and the computation time till \( t = 1000s \). We can see from the Fig. 5.2 that although these multi-symplectic schemes are not energy-conserving or momentum-conserving, they still can keep the error of global energy and momentum be bounded in a limited range. Furthermore, we can see that the MS IV scheme performs best in preserving the invariants as it bound the global energy and momentum error till \( 10^{-5} \) while others are \( 10^{-4} \) or bigger. In Table 5.1, we list the computation time of these multi-symplectic schemes in the propagation of single soliton till \( t = 1000s \) and we find MS III is the most efficient.

5.2. Example of two-solitary waves

As an example of a two-solitary wave interaction, the initial condition is chosen as
\[ u(x, 0) = 6sech^2(x). \]  
(5.2)

The parameters are taken as \( \eta = 6, \mu = 1 \). The computations are done on the space interval \([-20, 20]\) and the boundary is taken to be periodic as well. We take \( N = 300 \), time step \( \Delta t = 0.002 \) and space step \( \Delta x = \frac{40}{300} \).

Table 5.1: The computation time of the multi-symplectic schemes in a single soliton simulation with \( \Delta t = 0.2s, t = 1000s \).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>MS I</th>
<th>MS II</th>
<th>MS III</th>
<th>MS IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation Time</td>
<td>65.2161</td>
<td>36.3772</td>
<td>15.9360</td>
<td>43.889</td>
</tr>
</tbody>
</table>
Fig. 5.2. The invariants-preserving property of the proposed multi-symplectic algorithms in a single soliton simulation till time = 1000s.

Fig. 5.3 shows the interaction of two-solitary waves obtained by the schemes MS III and MS IV and the profile of scheme MS III is presented in Fig. 5.4. We can easily find the two-solitary waves travel toward and collide and then leave each other without changing their own shapes. Using the other multi-symplectic schemes can get the similar results. The numerical
Fig. 5.3. The interaction of two solitary waves obtained by using the scheme **MS III** (left) and **MS IV** (right).

Table 5.2: The computation time of the multi-symplectic schemes in a two-solitary wave interaction with $\Delta t = 0.02s, t = 100s$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>MS I</th>
<th>MS II</th>
<th>MS III</th>
<th>MS IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation Time</td>
<td>65.2161</td>
<td>36.3772</td>
<td>15.9360</td>
<td>43.8899</td>
</tr>
</tbody>
</table>

results reveal that the collision generates no radiation which seems quite elastic. In addition, we also draw the global energy error and global momentum error of these proposed multi-symplectic schemes on two-solitary waves interaction in Fig. 5.5. Note that it is the same as the single soliton simulation that the error of invariants can be bounded even if the schemes do not preserve them. We also find that the scheme **MS IV** preserves the invariants best. The computation time is listed in Table 5.2 with $\Delta t = 0.02s, t = 100s$ and the scheme **MS III** is still the most efficient.

Fig. 5.4. The profiles of the propagation and interaction of two-solitary waves obtained by using the scheme **MS III**.
5.3. Example of multiple solitary waves

We consider the KdV equation (1.1) with initial condition

$$u(x, 0) = \cos(\pi x).$$  \hspace{1cm} (5.3)
The parameters are taken as $\eta = 1, \mu = 0.022$. The space interval is $[0, 2]$ and the boundary is also taken to be periodic. In numerical experiments, we take $N = 399$, $\Delta t = \frac{2 \times 10^{-5}}{\pi}$ and $\Delta x = \frac{2}{399}$.

The numerical solutions are presented in Fig. 5.6 when $\pi t = 9$, $\pi t = 19$, $\pi t = 75$. Fig. 5.6
Fig. 5.8. The LMCS methods with $N = 399$, $\Delta t = \frac{5 \times 10^{-5}}{\pi}$. (a) The global errors of discrete total energy by using LMCS. (b) The global errors of discrete total momentum by using LMCS.

(a)-(c) present the propagation of wave obtained by using the scheme MS I while (d)-(f) are got from the scheme ZK [2]. From the Fig. 5.6, we find the scheme ZK appears to oscillation at $\pi t = 19$ while the proposed scheme MS I succeeds to simulate the propagation of wave correctly in a very long time computing. Note that the figure of MS II, MS III and MS IV are similar to the scheme MS I and here we do not list them repeated because of the space limitation.

Figs. 5.7 and 5.8 display the global energy errors and the global momentum errors of the proposed local energy-conserving schemes (LECS I, II, III, IV) and local momentum-conserving schemes (LMCS I, II, III, IV), respectively. It is observed that the errors of invariants are bounded to the round-off error even over a long time, especially for the scheme (LECS I), (LMCS I) and (LMCS II). From the plots, it is obvious that these proposed methods can preserve the invariant precisely.

6. Conclusions

In this paper, we give a unified framework to derive the local structure-preserving algorithms for the KdV equation. In this framework, we construct eight multi-symplectic algorithms, eight local energy-conserving algorithms and eight local momentum-conserving algorithms, which are independent of the boundary conditions. The key of the framework is the concatenating method and we describe the construction process of some proposed schemes to illustrate how to use the concatenating method for constructing the structure-preserving algorithms. Moreover, we prove rigorously that the proposed algorithms are structure-preserving in theory. Furthermore, linear stability analysis and numerical experiments of the proposed schemes are given behind to support the results. We find most of the proposed schemes are new and have good numerical performance. In other words, the theory of the local structure-preserving algorithm can be naturally applied to integrate the KdV equation.

The method proposed in this paper can also be used to obtain the local structure-preserving algorithms for other equations, such as nonlinear Schrödinger equation, Camassa-Holm equation and the Kawahara equation. Moreover, future works are planed to study the two-dimensional problems as well.

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