A MODIFIED HSS ITERATION METHOD FOR SOLVING THE COMPLEX LINEAR MATRIX EQUATION \( AXB = C \)

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Abstract

In this paper, a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method for solving the complex linear matrix equation \( AXB = C \) has been presented. As the theoretical analysis shows, the MHSS iteration method will converge under certain conditions. Each iteration in this method requires the solution of four linear matrix equations with real symmetric positive definite coefficient matrices, although the original coefficient matrices are complex and non-Hermitian. In addition, the optimal parameter of the new iteration method is proposed. Numerical results show that MHSS iteration method is efficient and robust.

Key words: MHSS iteration method, HSS iteration method, Linear matrix equation.

1. Introduction

In this paper, we consider the following linear matrix equation:

\[
AXB = C,  \tag{1.1}
\]

where \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times n} \) and \( C \in \mathbb{C}^{n \times n} \) are given matrices. Assume that \( A, B \) and \( C \) are large and sparse matrices, and let \( A = W + iT, B = U + iV \), where \( W, T, U, V \in \mathbb{R}^{n \times n} \) are real symmetric matrices, and \( T, V \) are positive definite, \( W, U \) are positive semidefinite. Then the matrix \( A \) is non-Hermitian. As a special case of the coupled Sylvester equations

\[
\sum_{j=1}^{n} A_{ij} X_j B_{ij} = C_i \quad (i = 1, \ldots, m),
\]

the complex symmetric linear matrix equation (1.1) arises in many problems of scientific computation and engineering applications. Its exact solution problems and the least-squares problems have been discussed in the areas of stability of linear systems [28, 29], power systems [35], linear algebra [30], FFT-based solution of certain time-dependent PDEs [22]. Generally speaking,
the sizes of $A$ and $B$ are usually very large and how to effectively solve this kind of equations involving literally hundreds or thousands of variables is under research.

As well known, the complex linear matrix equation (1.1) is mathematically equivalent to the following linear systems of equations

$$Ax = f,$$

where $A = B^T \otimes A$, and the vectors $x$ and $f$ contain the concatenated columns of the matrices $X$ and $C$, respectively, with $\otimes$ being the Kronecker product and $B^T$ representing the transpose of the matrix $B$ [21]. However, it is a numerically poor way to determine the solution $X$ of the complex linear matrix equation (1.1), as numerically solving the linear system of equations (1.2) is quite costly and ill-conditioned.

With the application of Kronecker products, some algorithms have been proposed to compute the solution of the linear matrix equation (1.1), see, e.g., [4,5,23,24,37,41,48,49]. Moreover, some efficient methods have been presented to solve the linear and nonlinear matrix equations [36,44,45]. The HSS iteration method for non-Hermitian positive definite linear systems of equations was firstly proposed by Bai, et al. in [3], and then it was extended to other equations and conditions. We refer to [7-14,16-20,33,34,43] and the references therein. However, using the idea of HSS iteration method to solve matrix equation has not been investigated except for the work in [2,40,42,46,50,51]. In [6], Bai, et al. presented a modified HSS iteration method for complex symmetric linear systems of equations. In this paper, we used the similar idea for solving complex symmetric linear matrix equation $AXB = C$ and presented a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method. The linear matrix equation $AXB = C$ is solved iteratively without using the Kronecker product, but adopt a new inner-outer iteration strategy. Although we use an inner-outer iteration strategy, the new HSS iteration method still preserves the convergence property of the “one-level” HSS iteration method without showing the effect of the inner iteration. In the MHSS iteration method, only two linear sub-systems with real and symmetric positive definite coefficient matrices need to be solved at each step instead of the solution of the shifted skew-Hermitian sub-systems of the linear matrix equations with coefficient matrices $\alpha I + iT$ and $\beta I + iV$. Besides, the computation of $X^{(k+\frac{1}{2})}$ only needs real arithmetic, then the computation of the iterates $X^{(k+1)}$ requires a modest amount of complex arithmetic due to the fact that the right hand side in the corresponding system is complex. Both can be efficiently computed either exactly by a sparse Cholesky factorization or inexactly by a preconditioned conjugate gradient scheme.

Moreover, as the theoretical shows, the MHSS iteration method will converge to the unique solution of the linear matrix equation (1.1) under certain conditions. Theoretical analysis also shows that an upper bound on the contraction factor of the MHSS iteration method depends on the spectra of the Hermitian parts $W$ and $U$, but is independent on the the spectra of the matrices $T, V, A$ and $B$, or on the eigenvectors of the matrices $W, U, V, T, A$ and $B$. We can also give the optimal parameters which minimize the upper bound of the contraction factor.

In the remainder of this paper, a matrix sequence $\{Y^{(k)}\}_{k=0}^\infty \subseteq \mathbb{C}^{n \times n}$ is said to be convergent to a matrix $Y \in \mathbb{C}^{n \times n}$, if the corresponding vector sequence $\{y^{(k)}\}_{k=0}^\infty \subseteq \mathbb{C}^{n^2}$ is convergent to the corresponding vector $y \in \mathbb{C}^{n^2}$, where the vectors $y^{(k)}$ and $y$ contain the concatenated columns of the matrices $Y^{(k)}$ and $Y$, respectively. If $\{Y^{(k)}\}_{k=0}^\infty$ is convergent, then its convergence factor and convergence rate are defined as those of $\{y^{(k)}\}_{k=0}^\infty$, correspondingly. In addition, we use $\lambda(W)$, $\|W\|_2$ and $\|W\|_F$ to denote the spectrum, the spectral norm, and the Frobenius norm of the matrix $W \in \mathbb{C}^{n \times n}$, respectively. Note that $\|\cdot\|_2$ is also used to represent the 2-norm of
2. MHSS Iteration Method

In this section, by making use of the special structure of the coefficient matrices $A$ and $B$, we derive a modification of the HSS iteration method that was initially proposed in Bai [3]. The new splitting iteration method will be referred to as the MHSS iteration method. Besides, this iteration method consists of inner and outer iterations, and each step of the inner iteration is exactly computed by direct methods.

First we can split $A$ and $B$ as follows:

$$H(A) = \frac{1}{2}(A + A^*) = W, S(A) = \frac{1}{2}(A - A^*) = iT;$$

$$H(B) = \frac{1}{2}(B + B^*) = U, S(B) = \frac{1}{2}(B - B^*) = iV;$$

see, e.g., [1,3,5]. Meanwhile, the matrices $A$ and $B$ can be rewritten as

$$A = (\alpha I + W) + (iT - \alpha I) = (\alpha I + iT) + (W - \alpha I),$$

$$B = (\beta I + U) + (iV - \beta I) = (\beta I + iV) + (U - \beta I).$$

Then we can transform the linear matrix equation (1.1) into the following scheme by the idea of HSS iteration method:

$$(\alpha I + W)X^{(k+\frac{1}{2})}B = (\alpha I - iT)X^{(k)}B + C; \quad (2.1a)$$

$$(\alpha I + iT)X^{(k+1)}B = (\alpha I - W)X^{(k+\frac{1}{2})}B + C; \quad (2.1b)$$

where $\alpha$ is a prescribed positive parameter. We can also rewrite the second matrix equation of (2.1) into the following form

$$(\alpha I + T)XB = (\alpha I + iW)XB - iC.$$  

In fact, as the linear matrix equation (1.1) is equivalent to $-iAXB = -iC$, we can split the coefficient matrix $-iA = (\alpha I + T) - (\alpha I + iW)$. So Equations (2.1) can be rewritten as follows:

$$(\alpha I + W)X^{(k+\frac{1}{2})}B = (\alpha I - iT)X^{(k)}B + C; \quad (2.2a)$$

$$(\alpha I + T)X^{(k+1)}B = (\alpha I + iW)X^{(k+\frac{1}{2})}B - iC. \quad (2.2b)$$

In Section 3 of the paper, we will derive the conditions that guarantee the convergence of the above iteration and give the choice of the optimal parameter. We continue to split the matrix $B$ and iteratively solve each equation of (2.2), i.e., by splitting $B$ we iteratively solve the first equation of (2.2) by

$$(\alpha I + W)X^{(k+\frac{3}{2},i+\frac{1}{2})}(\beta I + U)$$

$$= (\alpha I + W)X^{(k+\frac{3}{2},i)}(\beta I - iV) + (\alpha I - iT)X^{(k)}B + C; \quad (2.3a)$$

$$(\alpha I + W)X^{(k+\frac{3}{2},i+1)}(\beta I + V)$$

$$= (\alpha I + W)X^{(k+\frac{3}{2},i+\frac{1}{2})}(\beta I + iU) - i(\alpha I - iT)X^{(k)}B - iC. \quad (2.3b)$$
Similarly, iteratively solve the second equation of (2.2) by
\[(\alpha I + T)X^{(k+1,i+\frac{1}{2})}(\beta I + U) = (\alpha I + T)X^{(k+1,i)}(\beta I - iV) + (\alpha I + iW)X^{(k+\frac{1}{2})}B - iC, \quad (2.4a)\]
\[(\alpha I + T)X^{(k+1,i+1)}(\beta I + V) = (\alpha I + T)X^{(k+1,i+\frac{1}{2})}(\beta I + iU) - i(\alpha I + iW)X^{(k+\frac{1}{2})}B - C. \quad (2.4b)\]

So we can get the MHSS iteration method for solving the linear matrix equation (1.1).

**MHSS iteration method**

Give an initial guess \(X^{(0)} \in \mathbb{C}^{n \times n}\), for \(k = 0, 1, 2, \cdots\) until \(\{X^{(k)}\}_{k=0}^\infty \subseteq \mathbb{C}^{n \times n}\) converges.

1. Approximate the solution of
\[\begin{align*}
(\alpha I + W)Z^{(k)}B &= R^{(k)}, \\
& \quad \text{with } R^{(k)} = C - AX^{(k)}B.
\end{align*}\]
Compute \(Z^{(k)}\) iteratively by
\[\begin{align*}
(\alpha I + W)Z^{(k,i+\frac{1}{2})}(\beta I + U) &= (\alpha I + W)Z^{(k,i)}(\beta I - iV) + R^{(k)}, \quad (2.5a) \\
(\alpha I + W)Z^{(k,i+1)}(\beta I + V) &= (\alpha I + W)Z^{(k,i+\frac{1}{2})}(\beta I + iU) + R^{(k)}. \quad (2.5b)
\end{align*}\]
There must be an \(i\) such that \(Z^{(k)} = Z^{(k,i+1)}\) makes the residual
\[P^{(k)} = R^{(k)} - (\alpha I + W)Z^{(k)}B\]
to satisfies
\[\|P^{(k)}\|_F \leq \varepsilon_k \|R^{(k)}\|_F.\]
Then compute
\[X^{(k+\frac{1}{2})} = X^{(k)} + Z^{(k)}.\]

In the process of solving the two equations of (2.5), we choose direct methods to compute the exact solutions.

2. Approximate the solution of
\[\begin{align*}
(\alpha I + T)Z^{(k+\frac{1}{2})}B &= R^{(k+\frac{1}{2})}, \\
& \quad \text{with } R^{(k+\frac{1}{2})} = -iC + iAX^{(k+\frac{1}{2})}B.
\end{align*}\]
Compute \(Z^{(k+\frac{1}{2})}\) iteratively by
\[\begin{align*}
(\alpha I + T)Z^{(k+\frac{1}{2},i+\frac{1}{2})}(\beta I + U) &= (\alpha I + T)Z^{(k+\frac{1}{2},i)}(\beta I - iV) + R^{(k+\frac{1}{2})}, \quad (2.6a) \\
(\alpha I + T)Z^{(k+\frac{1}{2},i+1)}(\beta I + V) &= (\alpha I + T)Z^{(k+\frac{1}{2},i+\frac{1}{2})}(\beta I + iU) + R^{(k+\frac{1}{2})}. \quad (2.6b)
\end{align*}\]
There must be an \(i\) such that \(Z^{(k+\frac{1}{2})} = Z^{(k+\frac{1}{2},i+1)}\) makes the residual
\[Q^{(k+\frac{1}{2})} = R^{(k+\frac{1}{2})} - (\alpha I + T)Z^{(k+\frac{1}{2})}B\]
to satisfy
\[ \|Q^{(k+\frac{1}{2})}\|_F \leq \eta_k \|R^{(k+\frac{1}{2})}\|_F. \]

Then compute
\[ X^{(k+1)} = X^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}. \]

In the course of solving the two equations of (2.6), we take direct methods to compute the exact solutions. Here, \( \varepsilon_k \) and \( \eta_k \) are given tolerances which are used to control the accuracy of the inner iterations.

**Remark 2.1.** To further improve the computational efficiency of the MHSS iteration method, we can also solve the two sub-problems (2.3) and (2.4) inexactly by utilizing certain effective iteration method. For example, the Smith’s method [38], the alternating direction implicit (ADI) method [25, 32, 47], the block successive over-relaxation (BSOR) [39], and the matrix splitting methods [31], which naturally results in the corresponding inexactly MHSS iteration method (see [15]).

### 3. Convergence Analysis

For discussing the convergence of the MHSS iteration method, we just need to discuss the convergence property of the two-step splitting iteration (2.2), as the convergence analysis of the MHSS iteration method is completely similar but a little complicated. First, we give the unconditional convergence property of the two-step splitting iteration (2.2) for \( A \).

**Lemma 3.1.** ([3, 40]) Let \( A, B, C \in \mathbb{C}^{n \times n} \), \( A = M_i - N_i \) (\( i = 1, 2 \)) be two splittings of matrix \( A \), and let \( X^{(0)} \) be a given initial matrix. If \( \{X^{(k)}\} \) is a two-step iteration sequence defined by
\[
\begin{align*}
M_1 X^{(k+\frac{1}{2})} B &= N_1 X^{(k)} B + C, \\
M_2 X^{(k+1)} B &= N_2 X^{(k+\frac{1}{2})} B + C,
\end{align*}
\]
with \( k = 1, 2, \ldots \), then
\[ X^{(k+1)} = M_2^{-1} N_2^{-1} N_1 X^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})CB^{-1}. \]

This iterative process can be rewritten in vector form as
\[ x^{(k+1)} = I \otimes (M_2^{-1} N_2^{-1} N_1) x^{(k)} + (B^{-T} \otimes M_2^{-1}(I + N_2 M_1^{-1})) \text{vec}(C). \quad (3.1)\]

Moreover, if the spectral radius \( \rho(I \otimes (M_2^{-1} N_2^{-1} N_1)) \) is less than 1, then \( \{X^{(k)}\} \) converges to \( X^* \in \mathbb{C}^{n \times n} \) for all \( X^{(0)} \in \mathbb{C}^{n \times n} \).

By making use of Theorem 2.2 in [3], we have the following theorem.

**Theorem 3.1.** Assume that \( A, B, C \in \mathbb{C}^{n \times n} \), and \( A, B \) are positive definite matrices. Let \( A = W + iT \), \( \alpha \) and \( \beta \) be positive constants. Then the iteration matrix of (2.2) is given by
\[ M(\alpha) = I \otimes ((\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)), \]
and its spectral radius \( \rho(M(\alpha)) \) is bounded by
\[ \sigma(\alpha) = \max_{\lambda_j \in \text{sp}(W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\alpha + \lambda_j}, \]
where $\text{sp}(W)$ is the spectral set of the matrix $W$. Therefore, it holds that

$$\rho(M(\alpha)) < \sigma(\alpha) \leq 1, \quad \forall \alpha > 0.$$  

**Proof.** By direct computations we have

$$\rho(M(\alpha)) = \rho((\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT))$$

$$= \rho((\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)(\alpha I + T)^{-1})$$

$$\leq \| (\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)(\alpha I + T)^{-1} \|_2$$

$$\leq \| (\alpha I + iW)(\alpha I + W)^{-1} \|_2 \| (\alpha I - iT)(\alpha I + T)^{-1} \|_2.$$  

Since the matrices $W$ and $T$ are real symmetric matrices, there exist orthogonal matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$P^TWP = \Lambda_W, \quad Q^TQT = \Lambda_T,$$

where

$$\Lambda_T = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n), \quad \Lambda_W = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

with $\lambda_j$ ($1 \leq j \leq n$) and $\mu_j$ ($1 \leq j \leq n$) being the eigenvalues of the matrices $W$ and $T$, respectively. By assumption, we know $\lambda_j \geq 0$ and $\mu_j > 0$, $j = 1, \cdots, n$.

Besides, based on the orthogonal invariance of the Euclidean norm $\| \cdot \|_2$, we can further get the following upper bound on $\rho(M(\alpha))$:

$$\rho(M(\alpha)) \leq \| (\alpha I + i\Lambda_W)(\alpha I + \Lambda_W)^{-1} \|_2 \| (\alpha I - i\Lambda_T)(\alpha I + \Lambda_T)^{-1} \|_2$$

$$= \max_{\lambda_j \in \text{sp}(W)} \left| \frac{\alpha + i\lambda_j}{\alpha + \lambda_j} \right| \cdot \max_{\mu_j \in \text{sp}(T)} \left| \frac{\alpha - i\mu_j}{\alpha + \mu_j} \right|$$

$$= \max_{\lambda_j \in \text{sp}(W)} \sqrt{\frac{\alpha^2 + \lambda_j^2}{\alpha + \lambda_j}} \cdot \max_{\mu_j \in \text{sp}(T)} \sqrt{\frac{\alpha^2 + \mu_j^2}{\alpha + \mu_j}}.$$  

Recalling that $\mu_j > 0$ holds for all $\mu_j \in \text{sp}(T)(1 \leq j \leq n)$, we see that

$$\sqrt{\frac{\alpha^2 + \mu_j^2}{\alpha + \mu_j}} < \alpha + \mu_j.$$  

It then follows that

$$\rho(M(\alpha)) < \max_{\lambda_j \in \text{sp}(T)} \sqrt{\frac{\alpha^2 + \lambda_j^2}{\alpha + \lambda_j}} = \sigma(\alpha).$$

Obviously, $\sigma(\alpha) \leq 1$ holds for any $\alpha > 0$. Therefore, the iteration scheme (2.2) will converge for any initial guess. \qed

Moreover, for the convergence property we have the following theorem.

**Theorem 3.2.** Assume that the assumptions of Theorem 3.1 hold. If $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n \times n}$ is an iteration sequence generated by the MHSS iteration method and if $X^* \in \mathbb{C}^{n \times n}$ is the exact solution of the complex linear matrix equation (1.1), then we have

$$\|X^{(k+1)} - X^*\|_{S(A)} \leq \left(\sigma(\alpha) + \omega\theta\varepsilon_k\right)\|X^{(k)} - X^*\|_{S(A)},$$

where the norm $\| \cdot \|_{S(A)}$ is defined as $\| Y \|_{S(A)} = \|(\alpha I + iT)Y\|_F$ for any matrix $Y \in \mathbb{C}^{n \times n}$ and the constants $\varrho, \theta$ and $\omega$ are given by

$$\varrho = \|(\alpha I + iT)(\alpha I + W)^{-1}\|_2, \quad \theta = \|B^T \otimes A(\alpha I + iT)^{-1}\|_2,$$

$$\sigma(\alpha) = \|(\alpha I - W)(\alpha I + W)^{-1}\|_2, \quad \omega = \|B^{-T}\|_2.$$
In particular, if
\[ (\sigma(\alpha) + \omega \theta \eta_k)(1 + \omega \varepsilon_k) \leq 1, \]
then the iteration sequence \( \{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n \times n} \) converges to \( \{X^*\} \subseteq \mathbb{C}^{n \times n} \), where \( \varepsilon_{\text{max}} = \max_k \{\varepsilon_k\} \) and \( \eta_{\text{max}} = \max_k \{\eta_k\} \).

**Proof.** By making use of the Kronecker product and the notation introduced in Theorem 3.1, we can rewrite the above described MHSS iteration method in the following matrix-vector form:
\[
(B^T \otimes (\alpha I + W))z^{(k)} = r^{(k)},
\]
\[
(B^T \otimes (\alpha I + iT))z^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})},
\]
with \( r^{(k)} = \text{vec}(C) - (B^T \otimes A)\text{vec}(X^{(k)}) \), \( r^{(k+\frac{1}{2})} = \text{vec}(C) - (B^T \otimes A)\text{vec}(X^{(k+\frac{1}{2})}) \), where \( z^{(k)} \) and \( z^{(k+\frac{1}{2})} \) can make the residual
\[
p^{(k)} = r^{(k)} - (B^T \otimes (\alpha I + W))z^{(k)}
\]
to satisfy
\[
\|p^{(k)}\|_2 \leq \varepsilon_k \|r^{(k)}\|_2,
\]
and the residual
\[
q^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})} - (B^T \otimes (\alpha I + iT))z^{(k+\frac{1}{2})}
\]
to satisfy
\[
\|q^{(k+\frac{1}{2})}\|_2 \leq \eta_k \|r^{(k+\frac{1}{2})}\|_2.
\]
We denote \( M_1 = \alpha I + W, M_2 = \alpha I + iT, N_1 = \alpha I - iT \) and \( N_2 = \alpha I - W \). From the matrix-vector form of this method, we obtain
\[
x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}
\]
\[
= x^{(k)} + (B^T \otimes M_1)^{-1}(r^{(k)} - p^{(k)})
\]
\[
= (I \otimes M_1^{-1}N_1)x^{(k)} + (B^T \otimes M_1)^{-1}\text{vec}(C) - (B^T \otimes M_1)^{-1}p^{(k)},
\]
\[
x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}
\]
\[
= (I \otimes M_2^{-1}N_2)x^{(k+\frac{1}{2})} + (B^T \otimes M_2)^{-1}\text{vec}(C) - (B^T \otimes M_2)^{-1}q^{(k)}.
\]
We then obtain that
\[
x^{(k+1)} = \left(I \otimes M_2^{-1}N_2M_1^{-1}N_1\right)x^{(k)} + \left(B^T \otimes M_2^{-1}(I + N_2M_1^{-1})\right)\text{vec}(C)
\]
\[ - (B^T \otimes M_2)^{-1}\left(I \otimes N_2M_1^{-1}\right)p^{(k)} + q^{(k)}.
\]
If \( x^* \) is the exact solution of system (3.3), it satisfies
\[
x^* = (I \otimes M_1^{-1}N_1)x^* + (B^T \otimes M_1)^{-1}\text{vec}(C)
\]
\[ = \left(I \otimes M_2^{-1}N_2M_1^{-1}N_1\right)x^* + \left(B^{*-T} \otimes M_2^{-1}(I + N_2M_1^{-1})\right)\text{vec}(C).
\]
Then
\[
x^{(k+\frac{1}{2})} - x^* = (I \otimes M_1^{-1}N_1)(x^{(k)} - x^*) + (B^T \otimes M_1)^{-1}p^{(k)},
\]
\[
x^{(k+1)} - x^* = \left(I \otimes M_2^{-1}N_2M_1^{-1}N_1\right)(x^{(k)} - x^*) - (B^{*-T} \otimes M_2^{-1})\left(I \otimes N_2M_1^{-1}\right)p^{(k)} + q^{(k)}.
\]
So we can get the following inequalities:

\[
\begin{align*}
&\|x^{(k+\frac{1}{2})} - x^*\|_{M_2} \leq \|I \otimes (M_2 M_1^{-1} N_1 M_2^{-1})\|_2 \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \|B^{-T} \otimes M_2 M_1^{-1}\|_2 \|p^{(k)}\|_2, \\
&\|x^{(k+1)} - x^*\|_{M_2} \leq \|I \otimes (M_2^{-1} N_2 M_1^{-1} N_1)\|_{M_2} \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \|B^{-T} \otimes N_2 M_1^{-1}\|_2 \|p^{(k)}\|_2 + \|B^{-T}\|_2 \|q^{(k+\frac{1}{2})}\|_2, \\
&\|r^{(k)}\|_2 \leq \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2}, \\
&\|r^{(k+\frac{1}{2})}\|_2 \leq \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k+\frac{1}{2})} - x^*\|_{M_2}.
\end{align*}
\]

(3.4a) (3.4b) (3.4c) (3.4d)

According to the controlling conditions

\[
\|p^{(k)}\|_2 \leq \varepsilon_k \|r^{(k)}\|_2, \quad \|q^{(k+\frac{1}{2})}\|_2 \leq \eta_k \|r^{(k+\frac{1}{2})}\|_2,
\]

and (3.4), we have

\[
\begin{align*}
&\|p^{(k)}\|_2 \leq \varepsilon_k \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2}, \\
&\|q^{(k+\frac{1}{2})}\|_2 \leq \eta_k \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \varepsilon_k \|B^{-T} \otimes M_2 M_1^{-1}\|_2 \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2}, \\
&\|x^{(k+1)} - x^*\|_{M_2} \leq \|I \otimes (M_2^{-1} N_2 M_1^{-1} N_1)\|_2 \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \|B^{-T} \otimes N_2 M_1^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \|B^{-T}\|_2 \|\eta_k \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k)} - x^*\|_{M_2} \\
&\quad + \|B^{-T} \otimes M_2 M_1^{-1}\|_2 \|B^T \otimes A M_2^{-1}\|_2 \|x^{(k+\frac{1}{2})} - x^*\|_{M_2}.
\end{align*}
\]

After simple adjustment of above inequalities, we can get (3.2).

In actually computation, the selections of $\alpha$, $\beta$, $\varepsilon_k$ and $\eta_k$ should satisfy the condition that $(\sigma(\alpha) + \omega \theta \eta_k)(1 + \omega \theta \varepsilon_k)$ is less than 1 but does not need to approach zero. However, in order to achieve the best convergence rate, the selections of $\alpha$ and $\beta$ are based on the following discussions and conclusions, which are similar to the results in [3].

Under the assumptions of Theorem 3.1, we can get the optimal $\alpha$ by the following formula:

\[
\alpha^* = \arg \min_{\alpha} \left\{ \max_{\lambda_j \in \mathcal{Sp}(W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\alpha + \lambda_j} \right\} = \sqrt{\lambda_{\min}(W) \lambda_{\max}(W)}
\]

\[
\sigma(\alpha^*) = \frac{\sqrt{\kappa(W)} - 1}{\sqrt{\kappa(W)} + 1},
\]

where $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$ are the minimum and maximum eigenvalues of matrix $W$, respectively. $\kappa(W)$ is the spectral condition number of matrix $W$.

Besides, the optimal $\beta^*$ can be determined similarly by the following formula

\[
\beta^* = \arg \min_{\beta} \left\{ \max_{\mu_j \in \mathcal{Sp}(U)} \frac{\sqrt{\beta^2 + \mu_j^2}}{\beta + \mu_j} \right\} = \sqrt{\mu_{\min}(U) \mu_{\max}(U)}
\]

\[
\sigma(\beta^*) = \frac{\sqrt{\kappa(U)} - 1}{\sqrt{\kappa(U)} + 1}.
\]
where $\mu_{\text{min}}(U)$ and $\mu_{\text{max}}(U)$ are the minimum and maximum eigenvalues of matrix $U$, respectively. $\kappa(U)$ is the spectral condition number of matrix $U$.

In all, we know that the convergence rate of the MHSS iteration method only depends on the spectrum of the Hermitian symmetric positive semidefinite matrix parts $W$ and $U$, but does not depend on the the spectrums of the skew-Hermitian symmetric positive definite matrix parts $T$ and $V$, or the spectrum of the matrices $A$ and $B$, or the eigenvectors of the matrices $W$, $U$, $T$, $V$, $A$ and $B$. At the same time, we select the inner iteration tolerance $\varepsilon_k$ and $\eta_k$ to guarantee the convergence rate of the MHSS iteration method being as fast as possible. For the convergence rate, we can get the following theorem, which is straightforward according to Theorem 3.3 in [3].

**Theorem 3.3.** Assume that the conditions of Theorem 3.1 hold. Suppose that both $r_1(k)$ and $r_2(k)$ are nondecreasing and positive sequence satisfying $r_1(k) \geq 1$ and $r_2(k) \geq 1$, and $\lim_{k \to \infty} \sup r_1(k) = \lim_{k \to \infty} \sup r_2(k) = +\infty$, and that both $\delta_1$ and $\delta_2$ are real constants in the interval $(0, 1)$ satisfying

$$
\varepsilon_k \leq c_1 \delta_1^{r_1(k)} \quad \text{and} \quad \eta_k \leq c_2 \delta_2^{r_2(k)}, \quad k = 0, 1, \ldots
$$

where $c_1$ and $c_2$ are nonnegative constants. Then it holds that

$$
\|X^{(k+1)} - X^\star\|_{S(A)} \leq \left( \sqrt{\sigma(\alpha)} + \varphi \delta^{r(k)} \right)^2 \|X^{(k)} - X^\star\|_{S(A)},
$$

where $\rho$ and $\theta$ are defined in the above theorem and $r(k)$ and $\delta$ are defined as

$$
r(k) = \min\{r_1(k), r_2(k)\}, \quad \delta = \max\{\delta_1, \delta_2\},
$$

$$
\varphi = \max\left\{ \frac{\sqrt{c_1 c_2 \omega}}{2 \sqrt{\sigma(\alpha)}}, \frac{1}{c_1 \sigma(\alpha) + c_2 \omega} \right\}.
$$

In particular, we have

$$
\lim_{k \to \infty} \sup \frac{\|X^{(k+1)} - X^\star\|_{S(A)}}{\|X^{(k)} - X^\star\|_{S(A)}} = \sigma(\alpha).
$$

Certainly, we can make $\eta_k$ and $\varepsilon_k$ approach to zero with other ways, but we will not discuss here.

### 4. Numerical Examples

In this section, we give some examples to show the performance of the MHSS iteration method for solving the linear matrix equation (1.1). The numerical experiments are executed in Matlab on an Inter dual core processor (2.30GHz, 2GB RAM). All iterations of this section are started from zero matrix and stopped once the current residual norm satisfying

$$
\|R^{(k)}\|_F / \|R^{(0)}\|_F \leq 10^{-6},
$$

where $R^{(k)} = C - AX^{(k)}B$.
Table 4.1: IT and CPU for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>n</th>
<th>IT_out</th>
<th>IT_in</th>
<th>IT_in CPU RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>0.015 1.5699 × 10⁻⁷</td>
</tr>
<tr>
<td>16</td>
<td>9</td>
<td>4</td>
<td>4</td>
<td>0.046 4.9617 × 10⁻⁷</td>
</tr>
<tr>
<td>32</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0.156 5.9182 × 10⁻⁷</td>
</tr>
<tr>
<td>64</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0.531 6.3818 × 10⁻⁷</td>
</tr>
<tr>
<td>128</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>3.463 5.3250 × 10⁻⁷</td>
</tr>
<tr>
<td>256</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>27.549 3.9108 × 10⁻⁷</td>
</tr>
</tbody>
</table>

Table 4.2: The optimal values $\alpha_{exp}$, $\beta_{exp}$ for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>n</th>
<th>$\alpha_{exp}$</th>
<th>$\beta_{exp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>9.595</td>
<td>7.219</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>8.601</td>
<td>7.211</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>10.425</td>
<td>6.399</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>10.606</td>
<td>6.239</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>10.596</td>
<td>7.233</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>10.571</td>
<td>7.054</td>
<td></td>
</tr>
</tbody>
</table>

Example 4.1. To get the sparse matrices $A$ and $B$, we build them in the following structures:

$$A = \begin{pmatrix}
(10 + i) & 1 & \gamma \\
2 & (10 + i) & 1 \\
\vdots & \ddots & \ddots \\
\gamma & 2 & (10 + i)
\end{pmatrix},$$

$$B = \begin{pmatrix}
(8 + i) & 1 & \gamma \\
3 & (8 + i) & 1 \\
\vdots & \ddots & \ddots \\
\gamma & 3 & (8 + i)
\end{pmatrix},$$

$$C = (i + 1) * I + pp^T,$$

where $I$ is the identity matrix and $p = (1, 2^{\frac{1}{2}}, 3^{\frac{1}{2}}, \ldots, m^{\frac{1}{2}})^T$.

The computing results of the MHSS iteration method are listed in Tables 4.1 and 4.3, and the optimal parameters $\alpha_{exp}$, $\beta_{exp}$ are listed in Tables 4.2 and 4.4. When $\gamma = 0$, the results are listed in Tables 4.1 and 4.2 with $\varepsilon_k = 0.01$ and $\eta_k = 0.01$. When $\gamma = 1$, the results are listed in Tables 4.3 and 4.4.

Example 4.2. The linear matrix equation (1.1) with $m = n$ and the matrices

$$A = \text{diag}(1, 2, \ldots, n) + rL^T + i * I,$$

$$B = 2^{-i}I + \text{diag}(1, 2, \ldots, n) + rL^T + 2^{-i}L + i * I,$$
Table 4.3: IT and CPU for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>n</th>
<th>IT_out</th>
<th>IT_in</th>
<th>CPU</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>0.203</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0.546</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>3.198</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>24.835</td>
</tr>
</tbody>
</table>

Table 4.4: The optimal values $\alpha_{exp}$, $\beta_{exp}$ for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>$\alpha_{exp}$</th>
<th>$\beta_{exp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>10.259</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>9.870</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>9.550</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>10.545</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>10.542</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>10.541</td>
</tr>
</tbody>
</table>

Table 4.5: IT and CPU for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>n</th>
<th>IT_out</th>
<th>IT_in</th>
<th>IT_in</th>
<th>CPU</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>0.031</td>
<td>5.0970 $\times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>23</td>
<td>5</td>
<td>5</td>
<td>0.156</td>
<td>7.5423 $\times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>32</td>
<td>6</td>
<td>6</td>
<td>0.967</td>
<td>8.2125 $\times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>41</td>
<td>8</td>
<td>8</td>
<td>6.521</td>
<td>9.6446 $\times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>68</td>
<td>15</td>
<td>22</td>
<td>134.581</td>
<td>9.3735 $\times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 4.6: The optimal values $\alpha_{exp}$, $\beta_{exp}$ for MHSS.

<table>
<thead>
<tr>
<th>MHSS</th>
<th>n</th>
<th>$\alpha_{exp}$</th>
<th>$\beta_{exp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>2.607</td>
<td>5.240</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>4.590</td>
<td>5.239</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>7.425</td>
<td>6.399</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>10.601</td>
<td>8.211</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>10.586</td>
<td>11.231</td>
</tr>
</tbody>
</table>

where $L$ a strictly lower triangular matrix having ones in the lower triangle part and $t$ is a problem parameter to be specified in actual computations.

By the MHSS iteration method with $\varepsilon_k = 0.01$ and $\eta_k = 0.01$, we can get the computational results which are listed in Table 4.5 and the optimal parameters $\alpha_{exp}$, $\beta_{exp}$ are listed in Table 4.6.

**Remark 4.1.** When the coefficient matrices are complex, the GI method proposed in [26, 27]
always fail to obtain the solution.

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A Modified HSS Iteration Method for Solving Matrix Equation


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