Doubly Perturbed Neutral Stochastic Functional Equations Driven by Fractional Brownian Motion

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Abstract. In this paper, we study a class of doubly perturbed neutral stochastic functional equations driven by fractional Brownian motion. Under some non-Lipschitz conditions, we will prove the existence and uniqueness of the solution to these equations by providing a semimartingale approximation of a fractional stochastic integration.

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1 Introduction

As the limit process from a weak polymer model, the following doubly perturbed Brownian motion

$$X_t = B_t + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,$$

(1.1)

was presented in Norris, Rogers and Williams [1], also as the scaling limit of some self-interacting random walks [2], has attracted much interest from several directions, see Le Gall and Yor [3], Davis [4, 5], Carmona, Petit and Yor [6], Perman and Werner [7], Chaumont and Doney [8, 9], Werner [10], etc.

Following them, Doney and Zhang [11] have studied the following single perturbed stochastic functional equation:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + a \max_{0 \leq s \leq t} X_s,$$

(1.2)

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with the condition that $\sigma, b$ are Lipschitz continuous functions.

Recently, under some non-Lipschitz conditions, Luo [12] obtained the existence and uniqueness of the solution to the following doubly stochastic functional equation

$$X_t = X_0 + \int_0^t b(s,X_s) \, ds + \int_0^t \sigma(s,X_s) \, dB_s + \int_0^t \int_{-1}^{0} h(X_{s-},y) \tilde{N}(ds,dy) + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,$$

(1.3)

Hu and Ren [13] studied the existence and uniqueness of the solution to the following doubly perturbed neutral stochastic functional equation

$$X_t = X_0 + G(t,X_t) - G(0,X_0) + \int_0^t f(s,X_s) \, ds + \int_0^t \sigma(s,X_s) \, dB_s + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,$$

(1.4)

and Liu and Yang [14] proved the existence and uniqueness of the solution to the following doubly perturbed neutral stochastic functional equation with Markovian switching and Poisson jumps

$$X_t = X_0 + G(X_t,r(t)) - G(0,r_0) + \int_0^t f(s,r(s),X_s) \, ds + \int_0^t \sigma(s,r(s),X_s) \, dB_s + \int_0^t \int_{-\tau}^{0} h(X_{s-},y) \tilde{N}(ds,dy) + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s.$$

(1.5)

One solution for many SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [15], there is an increasing interest in stochastic models based on the fractional Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{ B^H(t), t \geq 0 \}$ with the covariance function

$$R_H(t,s) = \mathbb{E}(B^H_t B^H_s) = \frac{1}{2} (t^2H + s^2H - |t-s|^{2H}).$$

When $H = 1/2$ the fBm becomes the standard Brownian motion, and the fBm $B^H$ neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm $B^H$, $H > 1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property make fBm as a potential candidate to model noise in mathematical finance (see [16]); in biology (see [17]); in communication networks (see, e.g., [18]); the analysis of global temperature anomaly [19] electricity markets [20] etc.

In [15], Mandelbrot et al. have given a representation of $B^H_t$ of the form:

$$B^H_t = \frac{1}{\Gamma(1+\alpha)} \left( U(t) + \int_0^t (t-s)^\alpha \, dW_s \right),$$
where $\alpha = H - 1/2$, $U(t)$ is a stochastic process of absolutely continuous trajectories, and $W_t^H := \int_0^t (t-s)^\alpha dW_s$ is called a fBm of the Liouville form (LfBm). Because a LfBm shares many properties of a fBm (except that it has non-stationary increments) and for simplicity we use $W_t^H$ standing for $B_t^H$ throughout this paper.

The aim of this paper is to study the existence and uniqueness of solution of the following doubly perturbed neutral stochastic functional equation

$$X_t = X_0 + G(t, X_t) - G(0, X_0) + \int_0^t f(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + \int_0^t g(s) dW^H_s + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,$$

(1.6)

where $W_t^H$ is a LfBm with $H > 1/2$, $B_t$ is a standard Brownian motion independent of $W_t^H$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $G : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are appropriate mappings specified later.

The rest of this paper is organized as follows. In Section 2, we recall the definition of a stochastic integral with respect to LfBm from an approximate approach. Section 3 is devoted to giving the main results of the paper. An example is presented in Section 4 to illustrate the theory.

2 Preliminaries

In the last few decades, many differential ways have been introduced to constructed the fractional stochastic calculus (see, for instance, [21]). The main difficulties in studying fractional stochastic systems are that we cannot apply stochastic calculus developed by Itô since fBm is neither a Markov process nor a semimartingale, except for $H = 1/2$. Recently, an approximate approach has been developed to avoid those difficulties (see, [22, 23] and the references therein). Let us recall some fundamental results about this approach.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For every $\varepsilon > 0$ we define

$$W_t^{H, \varepsilon} := \int_0^t (t-s+\varepsilon)^\alpha dW_s.$$

In [22], author proved that $W_t^{H, \varepsilon}$ is a semimartingale with the following decomposition

$$W_t^{H, \varepsilon} = \varepsilon^\alpha W_t + \int_0^t q^\varepsilon(s) ds,$$

(2.1)

where $q^\varepsilon(s) = \int_s^0 a(s+\varepsilon-u)^{\alpha-1} dW_u$. Moreover, $W_t^{H, \varepsilon}$ converges to $W_t^H$ in $L^p(\Omega)$, $p > 1$ uniformly in $t \in [0, T]$ as $\varepsilon \rightarrow 0$:

$$\mathbb{E} |W_t^{H, \varepsilon} - W_t^H|^p \leq c_{p, T} \varepsilon^{\beta H}.$$
For $f$ is a deterministic function in $L^2[0,T]$, from the decomposition (2.1) we have

$$
\int_0^t f(s) dW^H_s = \int_0^t e^\alpha f(s) dW_s + \int_0^t \int_0^s \alpha f(s)(s+\varepsilon-u)^\alpha - 1 dW_u ds
$$

$$
= \int_0^t e^\alpha f(s) dW_s + \int_0^t \int_s^t \alpha f(u)(u+\varepsilon-s)^\alpha - 1 du dW_s. \quad (2.2)
$$

As $\varepsilon \to 0$, each term in the right-hand side of (2.2) converges in $L^2(\Omega)$ to the same term where $\varepsilon = 0$. Then, it is ‘natural’ to define (we can refer the reader to [24, 25] for a general definition).

**Definition 2.1.** For $f$ is a deterministic function in $L^2[0,T]$. The stochastic integral of $f$ with respect to LfBm is defined by

$$
\int_0^t f(s) dW^H_s := \lim_{\varepsilon \to 0} \int_0^t f(s) dW^H_s = \alpha \int_0^t \int_0^s f(u)(u-s)^\alpha - 1 du dW_s. \quad (2.3)
$$

**Lemma 2.1.** For $f$ is a deterministic function in $L^2[0,T]$. We can obtain the following estimate for the integral (2.3):

$$
\mathbb{E} \left( \max_{0 \leq s \leq t} \left| \int_0^s f(u) dW^H_u \right|^2 \right) \leq 4t^2 \alpha \int_0^t f^2(u) du. \quad (2.4)
$$

**Proof.** By applying Hölder’s inequality and Burkholder’s inequality we have

$$
\mathbb{E} \left( \max_{0 \leq s \leq t} \left| \int_0^s f(u) dW^H_u \right|^2 \right) \leq 4t^2 \alpha \int_0^t \left( \int_0^s f(u)(u-s)^\alpha - 1 \right)^2 du ds
$$

$$
\leq 4t^2 \alpha \int_0^t \left( \int_s^t f^2(u)(u-s)^\alpha - 1 du \right) \left( \int_0^s (u-s)^\alpha - 1 du \right) ds
$$

$$
= 4t^2 \alpha \int_0^t \left( \int_s^t f^2(u)(u-s)^\alpha - 1 du \right) ds
$$

$$
= 4t^2 \alpha \int_0^t \left( \int_0^s f^2(u)(u-s)^\alpha - 1 du \right) du
$$

$$
= 4t^2 \alpha \int_0^t f^2(u) du \leq 4t^2 \alpha \int_0^t f^2(u) du. \quad \square
$$

In order to obtain the existence and uniqueness of the solution to Eq. (1.6), we make the following assumptions:

(H1) There exists a function $A : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$
\mathbb{E} |f(t,X_t)|^2 + \mathbb{E} |\sigma(t,X_t)|^2 \leq A(t, \mathbb{E} \max_{0 \leq s \leq t} |X_s|^2),
$$
for all \( t \geq 0 \), where \( A(t,u) \) is locally integrable in \( t \) for each fixed \( u > 0 \) and is continuous nondecreasing in \( u \) for each fixed \( t \geq 0 \) and \( X_t : \mathbb{R}_+ \to \mathbb{R} \), and for any constant \( C \) the differential equation

\[
 u_t = u_0 + \int_0^t C A(s, u_s) \, ds,
\]

has a global solution.

(H2) There exists a function \( B : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
 \mathbb{E} |f(t, X_t) - f(t, Y_t)|^2 + \mathbb{E} |\sigma(t, X_t) - \sigma(t, Y_t)|^2 \leq B(t, \mathbb{E} \max_{0 \leq s \leq t} |X_s - Y_s|^2),
\]

for all \( t \geq 0 \) and \( X_t, Y_t : \mathbb{R}_+ \to \mathbb{R} \), where \( B(t,u) \) is locally integrable in \( t \) for each fixed \( u > 0 \) and is continuous nondecreasing in \( u \) for each fixed \( t \geq 0 \) and for any constant \( C \), if a non-negative function \( Y_t \) satisfies the following inequality:

\[
 Y_t \leq \int_0^t CB(s, Y_s) \, ds,
\]

for all \( t \geq 0 \), then \( Y_t \equiv 0 \).

(H3) There exists a positive constant \( K > 0 \) such that

\[
 |G(t, X_t) - G(t, Y_t)| \leq K |X_t - Y_t| \quad \text{and} \quad G(t,0) = 0,
\]

for all \( t \geq 0 \) and \( X_t, Y_t : \mathbb{R}_+ \to \mathbb{R} \).

(H4) \( K + |a| + |b| < 1 \).

3 Main results

The main results of our paper is the following theorem on the existence and uniqueness of the solution to the stochastic differential Eq. (1.6).

**Theorem 3.1.** Assume that the random variable \( X_0 \) is independent of \( B_t, W_t^H \), and \( \mathbb{E}(|X_0|^2) < \infty \), \( g \in L^2[0,T] \) and (H1)-(H4) hold. There exists a unique \( \mathcal{F}_t \)-adapted solution \( X_t, t \geq 0 \), to Eq. (1.6) such that \( \mathbb{E}(\max_{0 \leq s \leq T} |X_s|^2) < \infty \) for all \( T > 0 \).

**Proof.** We introduce the following iteration procedure. Let

\[
 X_t^0 = \frac{X_0}{1-a-b}, \quad 0 \leq t < \infty.
\]

For each integer \( n > 0 \), we define \( X^n \) as follows:

\[
 X_t^{n+1} = X_0 + G(t, X_t^{n+1}) - G(0, X_0) + \int_0^t f(s, X_s^n) \, ds + \int_0^t \sigma(s, X_s^n) \, dB_s \quad (3.1)
\]
+ \int_0^t g(s) \, dW_t^H + a \max_{0 \leq s \leq t} X^n_{s+1} + b \min_{0 \leq s \leq t} X^n_{s+1}. \quad (3.2)

In order to get the conclusion, we give three steps as follows:

**Step 1:** Let us show that \( \{ X^n_{t+1}, t \geq 0 \} \) is bounded. For any \( s \geq 0 \), we have

\[
|X^n_{s+1}| \leq |X_0| + |G(s,X^n_s)| + \left| \int_0^s f(u,X^n_u) \, du \right| + \left| \int_0^s \sigma(u,X^n_u) \, dB_u \right| \\
+ \left| \int_0^s g(u) \, dW_t^H \right| + |a| \max_{0 \leq u \leq s} |X^n_{u+1}| + |b| \min_{0 \leq u \leq s} |X^n_{u+1}|
\]

\[
\leq |X_0| + K|X^n_s| + \left| \int_0^s f(u,X^n_u) \, du \right| + \left| \int_0^s \sigma(u,X^n_u) \, dB_u \right| \\
+ \left| \int_0^s g(u) \, dW_t^H \right| + (|a| + |b|) \max_{0 \leq u \leq s} |X^n_{u+1}|. \quad (3.3)
\]

Taking the maximal value on both sides of (3.3), by Hölder’s inequality, the Burkholder’s inequality, (H1) and Lemma 2.1, we use \( C \) to denote a generic constant which may change from line to line in the rest of the work and get

\[
\mathbb{E}(\max_{0 \leq s \leq t} |X^n_{s+1}|^2) \leq C \left( \frac{1}{1 - |a| - |b|} \right)^2 \left[ \mathbb{E}|X_0|^2 + \int_0^t \mathbb{E}|f(s,X^n_s)|^2 \, ds + \int_0^t \mathbb{E}|\sigma(s,X^n_s)|^2 \, ds \right] \\
+ 4T^{2a} \int_0^t g^2(s) \, ds \leq C(\mathbb{E}|X_0|^2 + C \int_0^t \mathbb{E} \left[ |f(s,X^n_s)|^2 + |\sigma(s,X^n_s)|^2 \right] \, ds + \int_0^t \mathbb{E}|\sigma(s,X^n_s)|^2 \, ds \\
+ 4T^{2a} \int_0^T g^2(s) \, ds \leq C \left( \mathbb{E}|X_0| + \int_0^T g^2(s) \, ds \right) + C \int_0^t A(s,\mathbb{E} \max_{0 \leq u \leq s} |X^n_u|^2) \, ds. \quad (3.4)
\]

Owing to \( \mathbb{E}|X_0|^2 + \int_0^T g^2(s) \, ds < \infty \), we obtain that

\[
\mathbb{E}(\max_{0 \leq s \leq t} |X^n_{s+1}|^2) \leq Y_t \leq Y_T < \infty,
\]

where \( n = 0,1,2,\ldots \), and \( Y_t \) is a solution to the following equation:

\[
Y_t = Y_0 + \int_0^t C A(s,Y_s) \, ds,
\]

and then the boundedness of \( \{ X^n_{t+1}, t \geq 0 \} \) has been proved.

**Step 2:** Let us show that \( \{ X^n_{t+1}, t \geq 0 \} \) is Cauchy. For any \( s \geq 0, m,n \geq 0 \), we have

\[
|X^n_{s+1} - X^m_{s+1}| \leq |G(s,X^n_{s+1}) - G(s,X^m_{s+1})| + \left| \int_0^s f(u,X^n_u) \, du - \int_0^s f(u,X^m_u) \, du \right| \\
+ \left| \int_0^s \sigma(u,X^n_u) \, dB_u - \int_0^s \sigma(u,X^m_u) \, dB_u \right|.
\]
Owing to (H2), it is easy to get
\[
\begin{align*}
+ \left| \int_0^s \sigma(u, X^n_u) \, dB_u \right| + \left| \int_0^s \sigma(u, X^m_u) \, dB_u \right| \\
+ |a| \max_{0 \leq u \leq s} |X^n_u - X^m_u| + |b| \min_{0 \leq u \leq s} |X^n_u - X^m_u| \\
\leq K |X^n_{s+1} - X^m_{s+1}| + \int_0^s |f(u, X^n_u) - f(u, X^m_u)| \, du \\
+ \left| \int_0^s (\sigma(u, X^n_u) - \sigma(u, X^m_u)) \, dB_u \right| + (|a| + |b|) \max_{0 \leq u \leq s} |X^n_u - X^m_u|.
\end{align*}
\] (3.5)

Taking the maximal value on both sides of (3.5), by Hölder’s inequality, the Burkholder’s inequality and (H2), we can get
\[
\begin{align*}
\mathbb{E} \left( \max_{0 \leq s \leq t} |X^n_{s+1} - X^m_{s+1}|^2 \right) \leq & \mathcal{C} \left( \int_0^t \mathbb{E} |f(s, X^n_s) - f(s, X^m_s)|^2 \, ds \right) \\
& \quad + \int_0^t \mathbb{E} |\sigma(s, X^n_s) - \sigma(s, X^m_s)|^2 \, ds \\
\leq & \mathcal{C} \int_0^t B(s, \mathbb{E} \left( \max_{0 \leq u \leq s} |X^n_u - X^m_u|^2 \right)) \, ds. \quad (3.6)
\end{align*}
\]

Let
\[
Z_t := \limsup_{n,m \to \infty} \mathbb{E} \left( \max_{0 \leq s \leq t} |X^n_s - X^m_s|^2 \right).
\]

By (3.5) and the Fatou lemma, we can obtain
\[
\begin{align*}
Z_t &= \limsup_{n,m \to \infty} \mathbb{E} \left( \max_{0 \leq s \leq t} |X^n_s - X^m_s|^2 \right) \\
&\leq \limsup_{n,m \to \infty} \int_0^t B(s, \mathbb{E} \left( \max_{0 \leq u \leq s} |X^n_u - X^m_u|^2 \right)) \, ds \\
&\leq \mathcal{C} \int_0^t B(s, Z_s) \, ds.
\end{align*}
\]

Owing to (H2), it is easy to get
\[
\limsup_{n,m \to \infty} \mathbb{E} \left( \max_{0 \leq s \leq t} |X^n_s - X^m_s|^2 \right) \equiv 0,
\]
and then \( \{X^n_{t+1}, t \geq 0\} \) is Cauchy.

**Step 3:** Let us show that the solution to Eq. (1.6) is unique. Now suppose that \( X_t, t \geq 0, \) and \( Y_t, t \geq 0, \) are two solutions to Eq. (1.6), we have
\[
|X_t - Y_t| \leq \left| \int_0^t f(s, X_s) \, ds - \int_0^t f(s, Y_s) \, ds \right| + \left| \int_0^t \sigma(s, X_s) \, dB_s - \int_0^t \sigma(s, Y_s) \, dB_s \right|.
\]
\begin{align*}
&+ |G(t, X_t) - G(t, Y_t)| + |a| \max_{0 \leq s \leq t} X_s - \max_{0 \leq s \leq t} Y_s | + |b| \min_{0 \leq s \leq t} X_s - \min_{0 \leq s \leq t} Y_s |
\leq & K |X_t - Y_t| + \int_0^t |f(s, X_s) - f(s, Y_s)| \, ds \\
&+ \int_0^t (\sigma(s, X_s) - \sigma(s, Y_s)) \, dB_s | + (|a| + |b|) \max_{0 \leq s \leq t} |X_s - Y_s|. \quad (3.7)
\end{align*}

Taking the maximal value on both sides of (3.7), by Hölder’s inequality, the Burkholder’s inequality and (H2), we can get
\begin{align*}
\mathbb{E} \left( \max_{0 \leq s \leq t} |X_s - Y_s|^2 \right) \leq C \left( \frac{1}{1 - K - |a| - |b|} \right)^2 \left[ \int_0^t \mathbb{E} |f(s, X_s) - f(s, Y_s)|^2 \, ds \\
+ \int_0^t \mathbb{E} |\sigma(s, X_s) - \sigma(s, Y_s)|^2 \, ds \right] \\
\leq C \int_0^t B(s, \mathbb{E} \left( \max_{0 \leq u \leq s} |X_u - Y_u|^2 \right)) \, ds. \quad (3.8)
\end{align*}

By (H2), it is deduced that \( \mathbb{E} \left( \max_{0 \leq s \leq t} |X_s - Y_s|^2 \right) \equiv 0 \), then the solution to Eq. (1.6) is unique, and the proof is completed. \( \square \)

**Remark 3.1.** When \( g \equiv 0 \), Eq. (1.6) reduces to
\begin{align*}
X_t = & X_0 + G(t, X_t) - G(0, X_0) + \int_0^t f(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s \\
&+ a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s, \quad (3.9)
\end{align*}
which was recently studied in Hu and Ren [13], that is to say, Theorem 5 of [13] has been generalized.

### 4 An example

In this section, an example is provided to illustrate the obtained theory.

Consider the following doubly perturbed stochastic functional equation driven by fractional Brownian motion of the Liouville form:
\begin{align*}
X_t = & \int_0^t a X_s \, ds + \int_0^t b X_s \, dB_s + \int_0^t g(s) \, dW_s^H + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s, \quad (4.1)
\end{align*}
with the initial condition \( X_0 = \zeta \geq 0 \) (constant), where \( a, b, \alpha, \beta \) are constants and \( |a| + |b| < 1 \), \( g \in L^2[0, T] \). In order to get a unique \( \mathcal{F}_t \)-adapted solution \( X_t \), \( t \geq 0 \) to Eq. (4.1) by Theorem 3.1, let \( B(t, u) = \phi(t) \varphi(u) \) where \( \varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a continuous nondecreasing function such that
\( \varphi(0) = 0, \quad \int_0^\infty \frac{1}{\varphi(u)} \, du = +\infty, \)
\( \phi(t) \) is locally integrable. Here we present an example of such a function \( \varphi \). Define

\[
\varphi(u) = \begin{cases} 
    u \log(u^{-1}), & 0 \leq u \leq \varepsilon, \\
    \varepsilon \log(\varepsilon^{-1}) + \varphi'(\varepsilon-)(u-\varepsilon), & u > \varepsilon,
\end{cases}
\]

where \( \varepsilon > 0 \) is sufficiently small.

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