with the condition that $\sigma, b$ are Lipschitz continuous functions.

Recently, under some non-Lipschitz conditions, Luo [12] obtained the existence and uniqueness of the solution to the following doubly stochastic functional equation

$$
X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s + \int_0^t \int_{-1}^0 h(X_{s-}, y)N(ds, dy) + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,
$$

(1.3)

Hu and Ren [13] studied the existence and uniqueness of the solution to the following doubly perturbed neutral stochastic functional equation

$$
X_t = X_0 + G(t, X_t) - G(0, X_0) + \int_0^t f(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s,
$$

(1.4)

and Liu and Yang [14] proved the existence and uniqueness of the solution to the following doubly perturbed neutral stochastic functional equation with Markovian switching and Poisson jumps

$$
X_t = X_0 + G(X_t, r(t)) - G(0, r_0) + \int_0^t f(s, r(s), X_s)ds + \int_0^t \sigma(s, r(s), X_s)dW_s + \int_0^t \int_{-1}^0 h(X_{s-}, y)N(ds, dy) + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s.
$$

(1.5)

One solution for many SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [15], there is an increasing interest in stochastic models based on the fractional Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with the covariance function

$$
R_H(t, s) = \mathbb{E}(B^H_t B^H_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
$$

When $H = 1/2$ the fBm becomes the standard Brownian motion, and the fBm $B^H$ neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm $B^H$, $H > 1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property make fBm as a potential candidate to model noise in mathematical finance (see [16]); in biology (see [17]); in communication networks (see, for instance [18]); the analysis of global temperature anomaly [19] electricity markets [20] etc.

In [15], Mandelbrot et al. have given a representation of $B^H_t$ of the form:

$$
B^H_t = \frac{1}{\Gamma(1+\alpha)} \left( U(t) + \int_0^t (t-s)^\alpha dW_s \right),
$$

where $\alpha = H - 1/2$, $U(t)$ is a stochastic process of absolutely continuous trajectories, and $W^H_t := \int_0^t (t-s)^\alpha dW_s$ is called a fBm of the Liouville form (LfBm). Because a LfBm shares
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\[ + |G(t, X_t) - G(t, Y_t)| + |a| \max_{0 \leq s \leq t} X_s - \max_{0 \leq s \leq t} Y_s| + |b| \min_{0 \leq s \leq t} X_s - \min_{0 \leq s \leq t} Y_s| \]

\[ \leq K |X_t - Y_t| + \int_0^t |f(s, X_s) - f(s, Y_s)| ds \]

\[ + \int_0^t \rho(s, X_s - \sigma(s, Y_s)) d\beta_s + (|a| + |b|) \max_{0 \leq s \leq t} |X_s - Y_s|. \]  

(3.7)

Taking the maximal value on both sides of (3.6), by Hölder inequality, the Burkhölder inequality and (H2), we can get

\[ \mathbb{E}(\max_{0 \leq s \leq t} |X_s - Y_s|^2) \leq C \left( \frac{1}{1 - K - |a| - |b|} \right)^2 \left[ \int_0^t \mathbb{E}|f(s, X_s) - f(s, Y_s)|^2 ds \right] \]

\[ + \int_0^t \mathbb{E}|\sigma(s, X_s) - \sigma(s, Y_s)|^2 ds \]

\[ \leq C \int_0^t B(s, \mathbb{E}(\max_{0 \leq u \leq s} |X_u - Y_u|^2)) ds. \]  

(3.8)

By (H2), it is deduced that \( \mathbb{E}(\max_{0 \leq s \leq t} |X_s - Y_s|^2) \equiv 0 \), then the solution to Eq. (1.6) is unique, and the proof is completed. \( \square \)

**Remark 3.1.** When \( g \equiv 0 \), Eq. (1.6) reduces to

\[ X_t = X_0 + G(t, X_t) - G(0, X_0) + \int_0^t f(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s, \]  

(3.9)

which was recently studied in Hu and Ren [13], that is to say, Theorem 5 of [13] has been generalized.

### 4 An example

In this section, an example is provided to illustrate the obtained theory.

Consider the following doubly perturbed stochastic functional equation driven by fractional Brownian motion of the Liouville form:

\[ X_t = \int_0^t \alpha X_s ds + \int_0^t \beta X_s dB_s + \int_0^t g(s) dW^H_s + a \max_{0 \leq s \leq t} X_s + b \min_{0 \leq s \leq t} X_s, \]  

(4.1)

with the initial condition \( X_0 = \varepsilon \geq 0 \) (constant), where \( a, b, \alpha, \beta \) are constants and \( |a| + |b| < 1 \), \( g \in L^2[0, T] \). In order to get a unique \( F_t \)-adapted solution \( X_t \), \( t \geq 0 \) to Eq. (4.1) by Theorem 3.1, let \( B(t, u) = \phi(t) \varphi(u) \) where \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a continuous nondecreasing function such that \( \varphi(0) = 0 \) and \( \int_0^1 \frac{1}{\varphi(u)} du = +\infty \). \( \phi(t) \) is locally integrable. Here we present an example of such a function \( \varphi \). Define

\[ \varphi(u) = \begin{cases} 
  u \log(u^{-1}), & 0 \leq u \leq \varepsilon, \\
  e \log(e^{-1}) + \varphi'(u)(u - \varepsilon), & u > \varepsilon,
\end{cases} \]

where \( \varepsilon > 0 \) is sufficiently small.
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