Mixed Layer Problem of a Three-Dimensional Drift-Diffusion Model for Semiconductors

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Abstract. The quasineutral limit and the mixed layer problem of a three-dimensional drift-diffusion model is discussed in this paper. For the Neumann boundaries and the general initial data, the quasineutral limit is proven rigorously with the help of the weighted energy method, the matched asymptotic expansion method of singular perturbation problem and the entropy production inequality.

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1 Introduction

The drift-diffusion model is the most widely used model to describe semiconductor devices today [1]. It is a semi-classical macroscopic transport model and can be derived from the semiconductor Boltzmann equation [2]. The scaled three-dimensional bipolar drift-diffusion equations with the homogeneous Newmann boundary conditions and the general initial data read as follows:

\[ \partial_t n^\lambda - \text{div}(\nabla n^\lambda - n^\lambda \nabla V^\lambda) = 0, \quad x \in \Omega, \; t > 0, \]  
\[ \partial_t p^\lambda - \text{div}(\nabla p^\lambda + p^\lambda \nabla V^\lambda) = 0, \quad x \in \Omega, \; t > 0, \]  
\[ \lambda^2 \nabla V^\lambda = n^\lambda - p^\lambda - D(x), \quad x \in \Omega, \; t > 0, \]  
\[ \nabla n^\lambda \cdot \vec{n} = 0, \quad \nabla p^\lambda \cdot \vec{n} = 0, \quad \nabla V^\lambda \cdot \vec{n} = 0, \quad x \in \partial \Omega, \; t > 0, \]  
\[ n^\lambda(x,0) = n^\lambda_0(x), \quad p^\lambda(x,0) = p^\lambda_0(x), \quad x \in \overline{\Omega}. \]
where $\Omega = T^2 \times (0,1) \in \mathbb{R}^3$, the variables $n^\lambda, p^\lambda, V^\lambda$ are the electron density, the hole density and the electric potential, respectively. The constant $\lambda$ is the Debye length, the scaled physical parameter. The vector $\mathbf{n}$ is the unit outward normal vector along the boundary $\partial \Omega$. The doping profile $D = D(x)$ is a given function, and typically changes its sign. Here, we assume that $D(x)$ is a smooth function and the initial data $n_0^\lambda(x), p_0^\lambda(x)$ are smooth functions satisfying

$$\int_{\Omega} (n_0^\lambda(x) - p_0^\lambda(x) - D(x)) \, dx = 0.$$

Physically, the Debye length $\lambda^2 \approx 10^{-7} \ll 1$, and in this case the density of electrons almost equals to the density of holes, the zero Debye length limit $\lambda \to 0$ can be also called quasineutral limit. Letting $\lambda = 0$, we formally get the following quasineutral drift-diffusion model:

$$
\begin{align*}
\partial_t n_0^0 &= \text{div} (\nabla n_0^0 + n_0^0 \varepsilon^0), & x \in \Omega, \ t > 0, \\
\partial_t p_0^0 &= \text{div} (\nabla p_0^0 - p_0^0 \varepsilon^0), & x \in \Omega, \ t > 0, \\
0 &= n_0^0 - p_0^0 - D(x), & x \in \Omega, \ t > 0, \\
(\nabla n_0^0 + n_0^0 \varepsilon^0) \cdot \mathbf{n} \big|_{\partial \Omega} = 0, & x \in \partial \Omega, \ t > 0, \\
n_0^0(x,0) &= n_0^0(x), & x \in \Omega.
\end{align*}
$$

The initial data $n_0^0(x), p_0^0(x)$ are smooth functions satisfying the compatibility condition:

$$n_0^0 - p_0^0 - D(x) = 0.$$

The aim of this paper is to justify rigorously the limit that $(n^\lambda, p^\lambda, -\nabla V^\lambda) \to (n_0^0, p_0^0, \varepsilon^0)$ as $\lambda \to 0$.

The quasineutral limit problem is physically interesting (see [3]) and mathematically challenging (see [1,4]). Gasser et al. [5] studied the quasineutral limit of the time-dependent bipolar drift-diffusion-Poisson system, where no rigorous results on the quasineutral limit had been available before 2001. Jüngel and Peng [6] focused on the boundary layer problem of drift-diffusion-Poisson system in a bounded domain with mixed Dirichlet-Neumann boundary condition and initial conditions. The limit of the vanishing Debye length in a non-linear bipolar drift-diffusion model for semiconductors without a PN-junction was studied in [7]. Schmeiser [8] considered the initial value problem and quasineutral problem of the drift-diffusion model with the sign unchanging doping profile. For sign-changing and smooth doping profile with “good” boundary conditions, the quasineutral limit is strictly proved by introducing a “density” transform and Liapunov-type function firstly [9]. Based on [9], Wang [10] considered the multi-dimensional model and Wang [4,11] studied the limits in the spatial mean square norm uniformly in time and super-norm uniformly in space. Ju et al. [12] was devoted to the justification of the initial layer problem of bipolar transient drift-diffusion models in the multi-dimensional space.
The boundary layer problem in a bipolar drift-diffusion model with physical contact-insulating boundary conditions was studied in one-dimensional case in [13]. For the general initial data and the Neumann boundary conditions, the mixed layer problem was studied in one space dimension in [14].

For the quasineutral limit of the other dynamical models, some results can be founded in [15–19], etc.

In this paper, we study the mixed layer problem and the quasineutral limit of the drift-diffusion model for semiconductors in the case of the general sign-changing and smooth doping profile in three-dimensional space. Compared to [14], the new difficulty lies in the multi-dimensional case. One requires more technique like Sobolev’s estimates for higher order derivations, (see (4.5), step 3 and step 4 below), to overcome the difficulties arising from the nonlinear terms in the error system (4.3)-(4.4).

The plan of this paper is arranged as follows. Section 2 reformulates the problem. In Section 3, we construct a multi-scale approximate solution to the system (2.2)-(2.5) and state the main results of this paper. In Section 4, with the help of Sobolev lemma, we derive the energy estimates for the proof of main theorem in this paper.

In this paper, we denote the vector function $E_\lambda = (E_\lambda^1, E_\lambda^2, E_\lambda^3) = (E_{h\lambda}^1, E_{h\lambda}^2)$ with $E_{h\lambda}^1 = (E_{h\lambda}^1, E_{h\lambda}^2)$. Also $f_i = \partial f_i / \partial x_i, i = 1,2,3$, $f_t = \partial f_t / \partial t$. For example, $E_0^0 + I_3, \xi$ denotes $\partial E_0^0 + I_3 / \partial \xi$, and $E_0^0 + I_3$ denotes the third component of the vector function $E_0^0$.

### 2 Equivalent problem

Physically, the negative gradient of the electric potential is the electric field. We denote $-\nabla V_\lambda = E_\lambda$, where $E_\lambda$ is the electric field. Introduce the new variable $z_\lambda$ by the following transformation:

$$z_\lambda = n_\lambda^\lambda + p_\lambda^\lambda.$$

So system (1.1)-(1.5) can be rewritten as follows:

\begin{align}
\partial_t z_\lambda &= \text{div}(\nabla z_\lambda + E_\lambda^\lambda D(x)) - \lambda^2 \text{div}(E_\lambda^\lambda \text{div} E_\lambda^\lambda), \quad x \in \Omega, \ t > 0, \\
-\lambda^2 \partial_t E_\lambda^\lambda &= \nabla (D(x) - \lambda^2 \text{div} E_\lambda^\lambda) + z_\lambda^\lambda E_\lambda^\lambda, \quad x \in \Omega, \ t > 0, \\
\nabla z_\lambda \cdot \vec{n} \Big|_{\partial \Omega} &= 0, \quad \nabla (D - \lambda^2 \text{div} E_\lambda^\lambda) \cdot \vec{n} \Big|_{\partial \Omega} = 0, \quad E_\lambda^\lambda \cdot \vec{n} \Big|_{\partial \Omega} = 0, \quad x \in \partial \Omega, \ t > 0, \\
\lambda^2(x,0) = z_\lambda^0(x), \quad E_\lambda(x,0) = E_\lambda^0(x), \quad x \in \Omega,
\end{align}

where

$$z_\lambda^0(x) = n_0^\lambda(x) + p_0^\lambda(x), \quad \text{div} E_0^\lambda(x) = \frac{1}{\lambda^2} [D(x) - (n_0^\lambda(x) - p_0^\lambda(x))].$$

And the initial data $E_0^\lambda$ satisfies the following compatibility condition:

$$E_0^\lambda \cdot \vec{n} \Big|_{\partial \Omega} = 0.$$
For the system (1.6)-(1.10), letting \( z^0 = n^0 + p^0 \), we can obtain the equivalent system as follows:

\[
\begin{align*}
\partial_t z^0 &= \text{div}(\nabla z^0 + D(x)\epsilon^0), & x \in \Omega, & t > 0, \\
0 &= \nabla D + z^0\epsilon^0, & x \in \Omega, & t > 0, \\
\left[ \nabla z^0 + \epsilon^0(D - \lambda^2 \text{div} \epsilon^0) \right] \cdot \mathbf{n} |_{\partial \Omega} &= 0, & x \in \partial \Omega, & t > 0, \\
\left[ \nabla (D - \lambda^2 \text{div} \epsilon^0) + z^0 \epsilon^0 \right] \cdot \mathbf{n} |_{\partial \Omega} &= 0, & x \in \partial \Omega, & t > 0, \\
\lambda^3(x,0) &= z^0_0(x), & x \in \partial \Omega.
\end{align*}
\]

So the equivalent problem is the convergence of the system (2.2)-(2.5) to the system (2.6)-(2.10) as \( \lambda \to 0 \).

### 3 Construction of the approximate solution and main result

We assume that the initial data satisfy the following relationship:

\[
(z^0_0(x), E^0_0(x))^T = (z^0_0(x) + z_{0R}(x), E^0_0(x) + E_{0R}(x))^T,
\]

where the remainders \( z_{0R}, E_{0R} \) may depend upon \( \lambda \).

We establish the solution \((z^\lambda, E^\lambda)\) of the system (2.2)-(2.5) under the form:

\[
\begin{align*}
z^\lambda &= z^0 + \sum_{i=0}^{2} \lambda^i \left[ f(z^i_+ + z^i_-) + g(z^i_+ + z^i_-) + z^i_0 \right] + z_{\text{R}}, \\
E^\lambda &= \epsilon^0 + f(E^0_+ + E^0_-) + g(E^0_+ + E^0_-) + E^0_0 + E_{\text{R}}.
\end{align*}
\]

The domain of variabilities \( T^2 \times (0,1) \times (0,T) \) can be divided into six zones: the inner zone, the left boundary layer, the right boundary layer, the initial layer, the left mixed layer, and the right mixed layer. The inner function \((z^0, \epsilon^0)(x,t)\) is independent of \( \lambda \); \((z^i_+, E^0_+)(x_h \frac{3}{\lambda^i}, t)(i = 0,1,2)\) are the left boundary layer functions near the plane \( x_3 = 0 \); \((z^i_-, E^0_-)(x_h \frac{1-\xi_3}{\lambda^i}, t)(i = 0,1,2)\) are the right boundary layer functions near the plane \( x_3 = 1 \); \((z^i_{\text{R}}, E^0_{\text{R}})(x_h \frac{1-\xi_3}{\lambda^i}, t)(i = 0,1,2)\) are the initial time layer functions near \( t = 0 \); \((z^i_{\text{R}}, E^0_{\text{R}})(x_h \frac{\xi_3}{\lambda^i}, t)(i = 0,1,2)\) are the boundary layer functions near \( x_3 = 0 \) and \( t = 0 \); and \((z^i_{\text{R}}, E^0_{\text{R}})(x_h \frac{1-\xi_3}{\lambda^i}, t)(i = 0,1,2)\) are the boundary layer functions near \( x_3 = 1 \) and \( t = 0 \). The functions \((z_{\text{R}}, E_{\text{R}})\) are remainders, and satisfy the initial data:

\[
(z_{\text{R}}, E_{\text{R}})^T(x,0) = (z_{0R}, E_{0R})^T(x).
\]

Here \( x_h = (x_1, x_2) \). Let \( \xi = x_3 / \lambda, \eta = 1 - x_3 / \lambda, \tau = t / \lambda^2 \), then \( \xi, \eta, \) and \( \tau \) are large scale variables, \( \lambda \) and \( \lambda^2 \) are the lengths of the boundary layers and initial layers, respectively.
We ensure that
\[ (z^i_+, E^0_+)(x_h, \xi_+, t) \to 0, \quad \text{as } \xi_+ \to +\infty, \quad \text{for } i = 0, 1, 2, \]
\[ (z^i_-, E^0_-)(x_h, \eta_-, t) \to 0, \quad \text{as } \eta_- \to +\infty, \quad \text{for } i = 0, 1, 2, \]
\[ (z^i_+, E^0_+)(x_h, \xi, t) \to 0, \quad \text{as } \tau \to +\infty, \quad \text{for } i = 0, 1, 2, \]
\[ (z^i_+, E^0_+)(x_h, \xi, t) \to 0, \quad \text{as } \xi_+ \to +\infty, \quad \text{or } \tau \to +\infty, \quad \text{for } i = 0, 1, 2, \]
\[ \lambda \to \infty. \]

The cut-off functions \( f(x_3) \) and \( g(x_3) \) are \( C^2 \) smooth functions satisfying:
\[ f(0) = g(1) = 1, \]
\[ f(1) = f'(1) = f''(1) = f'(0) = f''(0) = 0, \]
\[ g(0) = g'(0) = g''(0) = g'(1) = g''(1) = 0. \]

Now we discuss in detail the functions in the boundary layer, the initial layer and the mixed layer, respectively. For brevity, the boundary layers and the mixed layers near the the left boundary \( x_3 = 0 \) would be considered here, the case near the right boundary \( x_3 = 1 \) can be dealt with similarly. Near the left boundary, the solution of system (2.2)-(2.5) can be written as follows:
\[ z^\lambda = z^0 + z^0_{+} + z^0_{++} + z^0_{+++} + \lambda (z^1_{+} + z^1_{++} + z^1_{+++}) + \lambda^2 (z^2_{+} + z^2_{++} + z^2_{+++}) + z_R, \]
\[ E^\lambda = \epsilon^0 + E^0_{+} + E^0_{++} + E^0_{+++} + E^0_{R}. \]

Inserting (3.4) and (3.5) into (2.2)-(2.3), by simple deduction one gets
\[ \frac{1}{\lambda^2} (\partial_\tau z^0_{+} + \partial_\tau z^0_{++} + \partial_\tau z^0_{+++}) + \frac{1}{\lambda} (\partial_\tau z^1_{+} + \partial_\tau z^1_{++} + \partial_\tau z^1_{+++}) + \partial_\tau z^2_{+} + \partial_\tau z^2_{++} + \partial_\tau z^2_{+++} \]
\[ + \lambda \lambda_1 z^1_{+} + \lambda^2 \lambda_2 z^2_{+} + \lambda^2 \lambda_3 z^2_{++} + \lambda^2 \lambda_4 z^2_{+++} \]
\[ = \frac{1}{\lambda^4} (z^0_{+} + z^0_{++} + z^0_{+++} + D E^0_{+} + D E^0_{++} + D E^0_{+++}) + \Delta z^0 + \Delta h z^0_{+} \]
\[ + \Delta h z^0_{++} + \Delta h z^0_{+++} + \Delta h z^1_{+} + \Delta h z^1_{++} + \Delta h z^1_{+++} + \lambda^2 (\Delta h z^2_{+} + \Delta h z^2_{++} + \Delta h z^2_{+++}) \]
\[ + \Delta h z^2_{++} + \Delta h z^2_{+++} + \delta (D E^\lambda) + \delta (D E_R) + \Delta z^0, \]
\[ (E^0_{+} + E^0_{++} - \lambda^2 (E^0_{+} + E^0_{++})) - \lambda^2 E_{Rh} - \lambda^2 E_{Rh} - \lambda^2 E_{Rh}. \]
\[-\lambda^2 \nabla_h \text{div}(\varepsilon^0 + E_0^0) - \lambda^2 \nabla_h \text{div}_h (E_{+h}^0 + E_{+lh}^0) + z_R (\varepsilon^0_h + E_{+h}^0 + E_{+lh}^0) \]
\[-z^0 E_{Rh} - \lambda^2 \nabla_h \text{div}_h E_R + E_{Rh} [z^0_+ + z^0_+ + z^0_+ + \lambda (z^1_+ + z^1_+ + z^1_+)] + \lambda^2 (z^2_+ + z^2_+ + z^2_+) \],
(3.7)

\[-(E_{+i3}^0 + E_{+i3}^0) - \lambda^2 (\varepsilon^0_{i3} + E_{+i3}^0) - \lambda^2 E_{R3,i} \]
\[= [z^0_{+i} + z^0_{+i} + z^0_{+i} + \lambda (z^1_{+i} + z^1_{+i} + z^1_{+i}) + \lambda^2 (z^2_{+i} + z^2_{+i} + z^2_{+i})] (\varepsilon^0_{i3} + E_{i3}^0) \]
\[+ E_{+i3}^0 + E_{i3}^0] + z^0 (E_{+i3}^0 + E_{+i3}^0 + E_{+i3}^0) - \lambda (E_{+i3}^0 + E_{+i3}^0 + E_{+i3}^0) \]
\[-\lambda^2 \text{div}_{E_{i3}^0} (\varepsilon_{i3}^0 + E_{i3}^0) - \lambda^2 \text{div}_{h} (E_{+ih,h}^0 + E_{+ih,h}^0) + z_R (\varepsilon_{i3}^0 + E_{+i3}^0 + E_{+i3}^0) \]
\[+ E_{i3}^0] - z^0 E_{R3} - \lambda^2 \text{div}_{E_{R3,i}} + E_{R3} [z^0_+ + z^0_+ + z^0_+ + \lambda (z^1_+ + z^1_+ + z^1_+)] \]
\[+ \lambda^2 (z^2_+ + z^2_+ + z^2_+) \].
(3.8)

Then by comparing the coefficients of \(O(\lambda^k)\) in the inner zone, the boundary layers, the initial layer and the mixed layers, we can obtain the inner and the layer functions respectively.

In the inner zone, comparing the coefficient of \(O(\lambda^0)\), one obtains the inner function \((z^0, \varepsilon^0)\) satisfying the system (2.6)-(2.10).

For system (1.1)-(1.5), we recall the classical existence theory for the drift-diffusion-Poisson system, see e.g., [20] and [21].

**Proposition 3.1.** (Existence and uniqueness) Assume that \((n^0_0, p^0_0) \in (C^2)^2\) satisfies the compatibility conditions

\[\nabla n^0 \cdot \vec{n} = 0, \quad \nabla p^0 \cdot \vec{n} = 0, \quad \text{on } x \in \partial \Omega.\]
(3.9)

Then system (1.1)-(1.5) has a unique global classical solution \((n^\lambda, p^\lambda) \in C^{2,1}(\Omega \times [0, \infty)), V^\lambda \in C^{4,1}(\Omega \times [0, \infty))\).

In the left boundary layer, by collecting the \(O(\lambda^{-2})\) amplitude term of equation (3.6), one gets the following system:

\[z^0_{+i,h} = 0, \quad z^0_{+i,h} \bigg|_{\xi = 0} = 0, \quad z^0_{+i,h} \bigg|_{\xi \to +\infty} = 0.\]

So \(z^0_+ = 0\).

At the order \(\lambda^0\) of (3.7), one finds that \(E_h = 0\) is the trivial solution. At the order \(\lambda^1\) of (3.8), one gets the boundary layer system for \(E^0_{+i3}\):

\[-E^0_{+i3,L} + z^0 (x_h, 0, t) E^0_{+i3} = 0, \quad E^0_{+i3} \bigg|_{\xi \to +\infty} = 0, \quad E^0_{+i3} \bigg|_{\xi = 0} = -\varepsilon^0_{3} (x_h, 0, t).\]
Using the decay conditions at infinity, one gets that:

\[ E_{\pm 3}^0 = -\varepsilon_0^2(x_h,0,t) e^{-\sqrt{2^0(x_h,0,t)} \xi}. \]

Comparing the coefficient of \( O(\lambda^{-1}) \) of (3.6), one obtains the boundary layer function \( z_1^+ \) satisfying the system:

\[ z_{1,\xi}^+ + E_{\pm 3,\xi}^0 D(x_h,0) = 0, \quad \left( z_2^0 + z_1^1 \right)_{|\xi=0} = 0. \]

By using the decay conditions and

\[ E_{\pm 3}^0 = -\varepsilon_0^2(x_h,0,t) e^{-\sqrt{2^0(x_h,0,t)} \xi}, \]

one gets:

\[ z_1^1 = -\varepsilon_0^2(x_h,0,t) D(x_h,0) \frac{e^{-\sqrt{2^0(x_h,0,t)} \xi}}{\sqrt{2^0(x_h,0,t)}}. \]

Since \( z_1^1 \neq 0 \), we assume that \( z_2^1 = 0 \).

In the initial layer, comparing the coefficient of \( O(\lambda^{-2}) \) and \( O(\lambda^{-1}) \) of (3.6), combining the initial and boundary conditions, one get the following system:

\[ z_0^0_{|t=0}, \quad z_0^0_{|\tau \to +\infty} = 0, \quad z_1^1_{|\tau \to +\infty} = 0, \quad z_{1,\tau}^1 = 0, \quad z_0^0_{|\tau=0} = 0, \quad z_1^1_{|\tau=0} = 0. \]

So \( z_1^0 = z_1^1 = 0 \).

Comparing the coefficient of \( O(\lambda^0) \) of (3.7) and (3.8), one obtains the system for the initial function \( E_1^0 \) as follows:

\[ -E_1^0 = z_0^0(x,0) E_1^0, \quad E_1^0_{|\tau=0} = E_0^0 - \varepsilon_0^0. \]

So \( E_1^0 = (E_0^0 - \varepsilon_0^0) e^{-z^0(x,0) \tau} \).

Collecting the \( O(\lambda^0) \) amplitude terms of equation (3.6), one gets the following system:

\[ z_{1,\tau}^1 = \text{div}(E_1^0 D), \quad z_1^1_{|\tau=0} = 0. \]

Inserting \( E_1^0 = (E_0^0 - \varepsilon_0^0) e^{-z^0(x,0) \tau} \) into the above system and solving the above system, one obtains that

\[ z_1^1 = \text{div} \left[ D \left( \frac{E_0^0 - \varepsilon_0^0}{z^0(x,0)} - \frac{E_0^0 - \varepsilon_0^0}{z^0(x,0)} e^{-z^0(x,0) \tau} \right) \right]. \]

In the left mixed layer, consider the \( O(\lambda^{-2}) \) amplitude term of equation (3.6), one gets the following system:

\[ z_{+L,\tau}^0 = z_{+L,\xi}^1 \xi = 0, \quad z_{+L,\xi}^0_{|\xi=0} = 0, \quad z_{+L,\xi}^0_{|\xi \to +\infty} = 0, \quad z_{+L}^0_{|\tau=0} = 0. \]
So \( z_{+1}^{0} = 0 \).

Comparing the coefficient of \( O(\lambda^{0}) \) of (3.7) and (3.8), one obtains the system for the mixed function \( E_{1} \) as follows:

\[
\begin{align*}
-E_{+1h,\tau}^{0} &= z_{h}^{0}(x_{h},0,0)E_{+1h}^{0}, \quad E_{+1h}^{0}|_{\tau=0} = -E_{+h}^{0}|_{\tau=0}, \\
E_{+13,\tau}^{0} &= E_{+13,\xi_{3}^{0}}^{0} - z_{h}^{0}(x_{h},0,0)E_{+13}^{0}, \\
E_{+13}^{0}|_{\xi_{3}^{0}} &= \varepsilon_{3}^{13}(x_{h},0,0)e^{-z_{h}^{0}(x_{h},0,0)\tau}, \quad E_{+13}^{0}|_{\xi_{3}^{0} \rightarrow +\infty} = 0, \\
E_{+13,\xi_{3}^{0}}^{0}|_{\tau=0} &= -E_{+3}^{0}|_{\tau=0} = \varepsilon_{3}^{13}(x_{h},0,0)e^{-\sqrt{2\varepsilon_{3}^{0}(x_{h},0,0)\xi_{3}^{0}}}.
\end{align*}
\]

Since \( E_{+h}^{0} = 0, \) so \( E_{+1h}^{0} = 0. \) By the comparison principle of parabolic equations, one can get that

\[
|E_{+13}^{0}| \leq |\varepsilon_{3}^{13}(x_{h},0,0)|e^{-\frac{1}{2}\sqrt{2\varepsilon_{3}^{0}(x_{h},0,0)\xi_{3}^{0}}}e^{-\frac{1}{2}\varepsilon_{3}^{0}(x_{h},0,0)\tau}.
\]

At the order \( \lambda^{-1} \) of (3.6), one gets the boundary layer system for \( z_{+1}^{1} \):

\[
\begin{align*}
z_{+1,\tau}^{1} &= z_{+1,\xi_{3}^{1}}^{1} + E_{0}^{13,\xi_{3}^{1}}D(x_{h},0), \quad z_{+1,\xi_{3}^{1}}^{1}|_{\xi_{3}^{1}=0} = 0, \quad z_{+1}^{1}|_{\xi_{3}^{1} \rightarrow +\infty} = 0, \\
z_{+1,\xi_{3}^{1}}^{1}|_{\tau=0} &= \varepsilon_{3}^{0}(x_{h},0,0)D(x_{h},0)\sqrt{2\varepsilon_{3}^{0}(x_{h},0,0)\xi_{3}^{0}}.
\end{align*}
\]

Similar to \( E_{+13}^{0} \), one can obtain that \( z_{+1}^{1} \) is also decreasing exponentially with respect to \( (\xi_{3}, \tau) \). Then assume \( z_{+1}^{2} = 0 \).

**Remark:** if \( E_{0}^{0} = \varepsilon_{0}^{1} \), then \( z_{0}^{2} = E_{0}^{0} = \pi_{0}^{2} = 0. \) In this case, there is no initial layer and no mixed layer.

By using the standard asymptotic expansion method of the perturbation theory, one can obtain the functions in right layers, similarity.

In addition, the boundary layer and initial layer functions have the following properties (see [4]).

**Proposition 3.2.** (1) \( z_{+}^{0} = z_{-}^{0} = z_{+}^{2} = z_{-}^{2} = E_{+h}^{0} = E_{-h}^{0} = \pi_{+}^{2} = \pi_{-}^{0} = 0. \)

(2) Assume that the inner solution \((z_{0}^{0}, e_{0}^{0}) \in (C^{\infty})^{4}, \) then for any \( T > 0, \) there exists a positive constant \( M \) independent of \( \lambda \) such that

\[
\begin{align*}
||\partial_{\xi_{3}^{k}}^{0}\partial_{\xi_{3}^{k}}^{0}(z_{+}^{0}, E_{+}^{0}, \pi_{+}^{1}, \eta_{-}^{k} E_{-}^{0}, \pi_{-}^{1})||_{L_{T}^{0}(\Omega \times [0,T])} &\leq M, \\
||\partial_{\xi_{3}^{k}}^{0}\partial_{\xi_{3}^{k}}^{0}(z_{+}^{0}, E_{+}^{0}, \pi_{+}^{1}, \eta_{-}^{k} E_{-}^{0}, \pi_{-}^{1})||_{L_{T}^{0}([0,T]^{2} \times I_{3}(0,1))} &\leq M^{2},
\end{align*}
\]

for any nonnegative integers \( k_{j}, \ j = 1,2, \cdots, 6. \)
Proposition 3.3. (1) $z_1^0 = z_1^1 = v_1^0 = v_1^1 = \pi_1^0 = 0$.
(2) Assume that the inner solution $(\xi, \epsilon_0) \in (C^\infty)^4$, then for any $T > 0$, there exists a positive constant $M$ independent of $\lambda$ such that
\[\|\partial_t^k \left( z_l^1, v_l^1, \tau^k \left( \partial_t^{\infty} (z_l^1, v_l^1), \partial_t^{\infty} (E_l^1, \pi_l^1) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M,\]
\[\|\partial_t^k \left( z_l^2, v_l^2, \tau^k \left( \partial_t^{\infty} (z_l^2, v_l^2), \partial_t^{\infty} (E_l^2, \pi_l^2) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M\lambda^k,\]
for any nonnegative integers $k_j$, $j = 1, 2, \ldots, 7$.

Proposition 3.4. (1) $z_{-1}^0 = z_{-1}^1 = z_{+1}^2 = z_{-1}^2 = E_{+0}^0 = E_{-1}^0 = \pi_{+1}^0 = \pi_{-1}^0 = 0$.
(2) Assume that the inner solution $(\xi, \epsilon_0) \in (C^\infty)^4$, then for any $T > 0$, there exists a positive constant $M$ independent of $\lambda$ such that
\[\|\partial_t^k \left( z_l^1, v_l^1, \tau^k \left( \partial_t^{\infty} (z_l^1, v_l^1), \partial_t^{\infty} (E_l^1, \pi_l^1) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M,\]
\[\|\partial_t^k \left( z_l^2, v_l^2, \tau^k \left( \partial_t^{\infty} (z_l^2, v_l^2), \partial_t^{\infty} (E_l^2, \pi_l^2) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M\lambda^k,\]
\[\|\partial_t^k \left( z_l^3, v_l^3, \tau^k \left( \partial_t^{\infty} (z_l^3, v_l^3), \partial_t^{\infty} (E_l^3, \pi_l^3) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M\lambda^{2k},\]
\[\|\partial_t^k \left( z_l^4, v_l^4, \tau^k \left( \partial_t^{\infty} (z_l^4, v_l^4), \partial_t^{\infty} (E_l^4, \pi_l^4) \right) \right) \|_{L^\infty_t([0,T] \times \Omega)} \leq M\lambda^{3k},\]
for any nonnegative integers $l_j$, $j = 1, 2, \ldots, 7$.

The main result of this paper can be stated as follows:

Theorem 3.1. Assume that the initial data satisfies (3.1) and
\[\|z_0\|_{H^2(\Omega)} \leq M\lambda^3, \quad \|\partial_t^2 z_0\|_{L^2(\Omega)} \leq M\lambda, \quad \|E_0\|_{H^2(\Omega)} \leq M\lambda^3, \quad \|\partial_t^2 E_0\|_{L^2(\Omega)} \leq M, \quad \|\partial_t^2 E_0\|_{L^2(\Omega)} \leq M\lambda^{-1}, \quad \text{for any } \lambda \in (0, \lambda_0), \quad \text{for any } \delta \in (0, 1), \quad \text{where } T_0 \text{ is the maximal existence time of the classical solution to the limit system (2.6)-(2.10), and } (z_R, E_R) \text{ is given by (3.2) and (3.3).}

4 Energy estimates

From the Propositions 3.2, 3.3 and 3.4, one reestablishes the solution of system (2.2)-(2.5):
\[z^1 = z^0 + \lambda \left[ f(z^1_{+1} + z^1_{-1}) + g(z^1_{-1} + z^1_{+1}) \right] + \lambda^2 z^2 + z_R, \quad (4.1)\]


\[ E^\lambda = \epsilon^0 + f(E^0_+ + E^0_{+1}) + g(E^0_+ + E^0_{-1}) + E^0_1 + E_R. \]  

(4.2)

Inserting (4.1) and (4.2) into the system (2.2)-(2.5) and using the equations of the inner and layer functions, one obtains

\[ z_{RI} - \Delta z_R = \text{div}(DE_R) - \lambda^2 \text{div}(E_R \text{div} E_R) + H, \quad x \in \Omega, \ t > 0, \]  

(4.3)

\[ \lambda^2 E_{RI} - \lambda^2 \nabla (\text{div} E_R) + z^0 E_R = -z_R E_R + J, \quad x \in \Omega, \ t > 0, \]  

(4.4)

where

\[ H = H_{in} + H_B + H_I + H_M + H_{BI} + H_{IM} + H_{BM} + H_R, \]

\[ J = J_{in} + J_B + J_I + J_M + J_{BI} + J_{IM} + J_{BM} + J_R, \]

\[ K = K_{in} + K_B + K_I + K_M + K_{BI} + K_{IM} + K_{BM} + K_R, \]

and

\[ H_{in} = -\lambda^2 \text{div}(\epsilon^0 \text{div} \eta^0), \]

\[ H_B = -f \lambda z^1_{+t} - g \lambda z^1_{-t} + \Delta_h (f \lambda z^1_+ + g \lambda z^-_1) \]

\[ + \lambda (f''z^1_+ + \frac{1}{\lambda} z^1_{+t} + g''z^1_+ - g' \frac{1}{\lambda} z^1_{-t}) \]

\[ + \text{div}_h(DfE^0_+ + DgE^0_-)_h + D_{x_3} (fE^0_{+3} + gE^0_{-3}) \]

\[ + [D - D(x_h,0)]f' E^0_{+3} + [D - D(x_h,1)]g' E^0_{-3} \]

\[ + \frac{1}{\lambda} [f(D - D(x_h,0)) E^0_{+3} - g(D - D(x_h,1)) E^0_{-3}] \]

\[- \lambda^2 \text{div} [E^0 \text{div}(fE^0_+ + gE^0_-) + (fE^0_+ + gE^0_-) \text{div}^0 \]

\[ + fE^0_1 \text{div}(fE^0_+ + gE^0_- \text{div}(gE^0_-)), \]

\[ H_I = \Delta(\lambda^2 z^2_1) - \lambda^2 \text{div}(\epsilon^0 \text{div} E^0_1 + E^0_1 \text{div} E^0_1 + E^0_0 \text{div} E^1), \]

\[ H_M = \Delta_h (f \lambda z^1_{+1} + g \lambda z^1_{-1}) + \lambda (f''z^1_{+1} + 2f' \frac{1}{\lambda} z^1_{+1} + g''z^1_{-1} - 2g' \frac{1}{\lambda} z^1_{-1}) \]

\[ + \text{div}_h(DfE^0_+ + DgE^0_-)_h + D_{x_3} (fE^0_{+13} + gE^0_{-13}) \]

\[ + \frac{1}{\lambda} [f(D - D(x_h,0)) E^0_{+13} - g(D - D(x_h,1)) E^0_{-13}] + D(fE^0_{+13} + gE^0_{-13}) \]

\[- \lambda^2 \text{div} [E^0 \text{div}(fE^0_+ + gE^0_-) + fE^0_1 \text{div}(fE^0_+ + gE^0_-) \text{div}^0 \]

\[ + (fE^0_+ + gE^0_-) \text{div}^0], \]

\[ H_{BI} = -\lambda^2 \text{div} [fE^0_0 \text{div} E^0_1 + E^0_1 \text{div}^0(fE^0_+ + gE^0_-)], \]
\[ H_{IM} = -\lambda^2 \text{div} \left[ f E_{i+1} \text{div} E_0 + g E_{i-1} \text{div} E_0 + E_0 \text{div} (f E_{i+1} + g E_{i-1}) \right], \]
\[ H_{BM} = -\lambda^2 \text{div} \left[ f E_{i+1} \text{div} E_0 + f E_{i-1} \text{div} (f E_i + g E_0) + g E_{i-1} \text{div} (g E_0 + g E_{i-1}) \right], \]
\[ H_R = -\lambda^2 \text{div} \left[ f \text{div} E_R + (f E_{i+1} + f E_{i-1}) \text{div} E_R + (g E_0 + g E_{i-1}) \text{div} E_R + E_0 \text{div} E_R + E_0 \text{div} (E_0 + g E_{i+1} + g E_0 - g E_{i-1} + E_i) \right], \]
\[ J_{im} = -\lambda^2 \xi^i_e + \lambda^2 \nabla (\text{div} E^0) \]
\[ J_B = -\lambda^2 f E_{i+1} - \lambda^2 g E_{i-1} - (f \lambda z^1_+ + g \lambda z^-_1) E^0 \]
\[ - (z^0 - z^0(x_h,0,t) + f \lambda z^1_+) f E^0 - (z^0 - z^0(x_h,1,t) + g \lambda z^-_1) g E^0 + \lambda^2 \left( \nabla_h \left( f' E_{i+1} + f' E_{i-1} + g' E_0 - g' E_{i-1} \right) \right), \]
\[ J_I = -\lambda^2 z_0^2 f^0 - (z^0 - z^0(x,0)+\lambda z_0^2) E^0 - \lambda^2 \nabla (\text{div} E^0) \]
\[ J_{M} = - (f \lambda z^1_+ + g \lambda z^-_1) E^0 - (z^0 - z^0(x_h,0,0) + f \lambda z^1_+) f E^0 + (z^0 - z^0(x_h,1,0) + g \lambda z^-_1) g E^0 + \lambda^2 \left( \nabla_h \left( f' E_{i+1} + f' E_{i-1} + g' E_0 - g' E_{i-1} \right) \right), \]
\[ J_{BI} = -\lambda^2 z_0^2 (f E^0 + g E^0) - (f \lambda z^1_+ + g \lambda z^-_1) E^0, \]
\[ J_{IM} = -\lambda^2 z_0^2 (f E^0 + g E^0) - (f \lambda z^1_+ + g \lambda z^-_1) E^0, \]
\[ J_{BM} = -f \lambda z^1_+ f E^0 - f \lambda z^1_+ f E^0 + f \lambda z^1_+ g E^0 - g \lambda z^-_1 g E^0, \]
\[ J_R = - (e^0 + f E^0 + f E^0 + f E^0 + g E^0 + g E^0 + E^0) z_R \]
\[ - \left[ f \lambda (z^1_+ + z^-_1) + g \lambda (z^-_1 + z^1_1) + \lambda^2 z_0^2 \right] E_R. \]

By the boundary condition of the system (2.2)-(2.5) and the system (2.6)-(2.10), one gets:
\[ (E_R \cdot \vec{n}) |_{\partial \Omega} = 0, \]
\[ |D(f \cdot E_{i+1}) \cdot (0,0,1)| \xi=0 = 0, \]
\[ |D(g \cdot E_{i+1}) \cdot (0,0,1)| \eta=0 = 0, \]
\[ z_{R,x_3} |_{\partial \Omega} = 0, \]
\[ (E^a \cdot \vec{n}) |_{\partial \Omega} = 0. \]

One also needs the following technical lemmas, which can be found in [4] and [17].

**Lemma 4.1.** For \( f, g \in H^1(\Omega) \), we have
\[ \|f g\|_{L^2} \leq \|f\|_{L^4} \cdot \|g\|_{L^4} \leq C \|f\|_{H^1} \cdot \|g\|_{H^1}, \]
where \( \Omega = T^2 \times (0,1) \).
Lemma 4.2. Let \( \Gamma^\lambda(t), G^\lambda(t) \) be nonnegative functions satisfying:

\[
\Gamma^\lambda(t) + \int_0^t G^\lambda(s) \, ds \leq M \Gamma^\lambda(t = 0) + M \int_0^t \left( \Gamma^\lambda(s) + (\Gamma^\lambda(s))^2 \right) \, ds
\]

\[
+ M \int_0^t \Gamma^\lambda(s) G^\lambda(s) \, ds + M(\Gamma^\lambda(t))^2 + M\lambda, \quad t \geq 0,
\]

(4.6)

where \( M \) is some positive constant independent of \( \lambda \). Then for any \( T \in [0, T_{\max}) \), \( T_{\max} \leq \infty \), there exists \( \lambda_0 \ll 1 \) such that for any \( \lambda \in (0, \lambda_0] \), if \( \Gamma^\lambda(t = 0) \leq \tilde{M} \lambda^{\min(a, 1)} \) for some \( a > 0 \), then

\[
\Gamma^\lambda(t) \leq \tilde{M} \lambda^{\min(a, 1) - \delta}
\]

holds for some constant \( \tilde{M} \) independent of \( \lambda \), and any \( \delta \in (0, \min(a, 1)) \) and \( 0 \leq t \leq T \). The inequality (4.6) is called an entropy production integration inequality.

With the help of the entropy production integration inequality and the energy method, one can prove the Main Theorem by the following six steps.

Proof. Step 1. The estimates on \( \|z_R, \lambda E_R\|_\Omega^2 \).

First, multiply (4.3) by \( z_R \), and integrate the resulting equation over \( \Omega \) with respect to \( x \), one gets

\[
\int_\Omega z_R \Delta z_R \, dx - \int_\Omega \Delta z_R \, dx = \int_\Omega \text{div}(D E_R) z_R \, dx - \lambda^2 \int_\Omega \text{div}(E_R \text{div} E_R) z_R \, dx + \int_\Omega H_{in} z_R \, dx
\]

\[
+ \int_\Omega H_B z_R \, dx + \int_\Omega H_I z_R \, dx + \int_\Omega H_M z_R \, dx + \int_\Omega H_{B_I} z_R \, dx
\]

\[
+ \int_\Omega H_{B_M} z_R \, dx + \int_\Omega H_{R} z_R \, dx.
\]

By the properties of the inner and layer functions, the Cauchy-Schwarz inequality, the inequality (4.5), the Sobolev lemma \((H^2(\Omega) \hookrightarrow L^\infty(\Omega))\), and integration by parts, one obtains

\[
\frac{d}{dt}\|z_R\|^2 + \|\nabla z_R\|^2 \leq M \|z_R\|^2 + M \|E_R\|^2 + M \lambda^4 \|\text{div} E_R\|^2 + M \lambda^4 \|E_R\|^2
\]

\[
+ M \lambda^4 \|E_R\|_{H^2} \|\text{div} E_R\| + M \lambda.
\]

(4.7)

Multiply (4.4) by \( E_R \), and integrate the resulting equation over \( \Omega \) with respect to \( x \), one gets

\[
\lambda^2 \int_\Omega E_R \cdot \dot{E}_R \, dx - \lambda^2 \int_\Omega \nabla (\text{div} E_R) : \dot{E}_R \, dx + \int_\Omega z^0 E_R \cdot \dot{E}_R \, dx
\]

\[
= - \int_\Omega z_R E_R \cdot \dot{E}_R \, dx + \int_\Omega J_{in} \cdot E_R \, dx + \int_\Omega J_B \cdot E_R \, dx + \int_\Omega J_I \cdot E_R \, dx
\]
By the properties of the inner and layer functions, the Cauchy-Schwarz inequality, the inequality (4.5), the Sobolev lemma \((H^2(\Omega) \hookrightarrow L^\infty(\Omega))\), \(z^0 \geq C_0 > 0\), and integration by parts, one obtains

\[
\frac{d}{dt} \lambda^2 \|E_R\|^2 + \lambda^2 \|\text{div}E_R\|^2 + C_0 \|E_R\|^2 \leq M \|z_R\|^2 + M \|z_R\|_{H^2}^2 \|E_R\|^2 + M \lambda. \tag{4.8}
\]

Then integrating \(\delta_1(4.7)+4.8\) with respect to \(t\), and taking \(\delta_1 > 0\) small enough, one obtains

\[
\|z_R\|^2 + \lambda^2 \|E_R\|^2 + \int_0^t \|\nabla z_R\|^2 dt + \int_0^t \|E_R\|^2 dt + \lambda^2 \int_0^t \|\text{div}E_R\|^2 dt \\
\leq \|z_R(x,0)\|^2 + \lambda^2 \|E_R(x,0)\|^2 + M \int_0^t \|z_R\|^2 dt + M \int_0^t \|z_R\|_{H^2}^2 \|E_R\|^2 dt \\
+ M \lambda^4 \int_0^t \|E_R\|_{H^2}^2 \|\text{div}E_R\|^2 dt + M \lambda. \tag{4.9}
\]

Step 2. The estimates on \(\|(z_{R,t}, \lambda E_{R,t})\|^2\).

Differentiating (4.3) with respect to \(t\), then multiplying the resulting equation by \(z_{R,t}\), and integrating the resulting equation over \(\Omega \times [0,t]\) with respect to \(x\) and \(t\), and applying integration by parts, one gets

\[
\int_0^t \int_\Omega z_{R,t}z_{R,t} dx dt - \int_0^t \int_\Omega \Delta z_{R,t}z_{R,t} dx dt \\
= \int_0^t \int_\Omega \text{div}(DE_{R,t})z_{R,t} dx dt - \lambda^2 \int_0^t \int_\Omega \text{div}(E_R \text{div}E_R)z_{R,t} dx dt + \int_0^t \int_\Omega H_{R}z_{R,t} dx dt. \tag{4.10}
\]

Differentiating (4.4) with respect to \(t\), then multiplying the resulting equation by \(E_{R,t}\), and integrating the resulting equation over \(\Omega \times [0,t]\) with respect to \(x\) and \(t\), and applying integration by parts, one gets

\[
\lambda^2 \int_0^t \int_\Omega E_{R,t} \cdot E_{R,t} dx dt - \lambda^2 \int_0^t \int_\Omega \nabla(\text{div}E_R)_{t} \cdot E_{R,t} dx dt + \int_0^t \int_\Omega z_0 E_{R,t} \cdot E_{R,t} dx dt \\
= -\int_0^t \int_\Omega (z_{R,E_R})_{t} \cdot E_{R,t} dx dt - \int_0^t \int_\Omega \tilde{z}_0 E_{R,t} \cdot E_{R,t} dx dt + \int_0^t \int_\Omega J_{I} \cdot E_{R,t} dx dt. \tag{4.11}
\]

For the equations (4.10) and (4.11), using the method similar in Step 1, one obtains

\[
\|z_{R,t}\|^2 + \lambda^2 \|E_{R,t}\|^2 + \int_0^t (\|\nabla z_{R,t}\|^2 + \lambda^2 \|\text{div}E_{R,t}\|^2 + C_0 \|E_{R,t}\|^2) dt \\
\leq \|z_{R,t}(x,0)\|^2 + \lambda^2 \|E_{R,t}(x,0)\|^2 + M \int_0^t (z_{R,t}, z_{R,t}, E_{R,t} \text{div}E_R)^2 dt
\]
\begin{equation}
+ M \int_0^t (\|z_{R_1}\|^2_{H^1} \|E_R\|^2_{H^1} + \|z_R\|^2_{H^2} \|E_{R_1}\|^2) dt \\
+ \int_0^t M \lambda^4 (\|\nabla E_{R_1}\|^2\|E_R\|^2_{H^1} + \|\nabla E_R\|^2_{H^1} \|E_{R_1}\|^2) dt + M \lambda.
\end{equation}

Step 3. The estimates on \(\|\nabla z_R, \lambda \nabla E_R\|\|^2\).

Multiplying (4.3) by \(-\Delta z_R\), multiplying (4.4) by \(-\Delta \nabla E_R\), and integrating the resulting equations over \(\Omega \times [0,t]\) with respect to \(x\) and \(t\), using the method similar in Step 1, one obtains

\begin{equation}
\|\nabla z_R\|^2 + \lambda^2 \|\nabla E_R\|^2 + \int_0^t \|\Delta z_R\|^2 dt + \lambda^2 \int_0^t \|\nabla \nabla E_R\|^2 dt + C_0 \int_0^t \|\nabla E_R\|^2 dt \\
\leq \|\nabla z_R(x,0)\|^2 + \lambda^2 \|\nabla E_R(x,0)\|^2 + M \int_0^t \|(E_R, z_{R_1}, \nabla z_R)\|^2 dt \\
+ M \int_0^t (\|\nabla z_R\|^2_{H^1} \|E_R\|^2_{H^1} + \|z_R\|^2_{H^2} \|\nabla E_R\|^2) dt \\
+ M \lambda^4 \int_0^t (\|\nabla E_R, \nabla \nabla E_R\|^2) dt + M \lambda.
\end{equation}

Step 4. The estimates on \(\|\nabla z_{R_1}, \lambda \nabla E_{R_1}\|\|^2\).

Multiplying (4.3) by \(-\Delta z_{R_1}\), multiplying (4.4) by \(-\Delta \nabla E_{R_1}\), and integrating the resulting equations over \(\Omega \times [0,t]\) with respect to \(x\) and \(t\), using the method similar in Step 1, one obtains

\begin{equation}
\|\nabla z_{R_1}\|^2 + \lambda^2 \|\nabla E_{R_1}\|^2 + \int_0^t \|\Delta z_{R_1}\|^2 dt + \lambda^2 \int_0^t \|\nabla \nabla E_{R_1}\|^2 dt + C_0 \int_0^t \|\nabla E_{R_1}\|^2 dt \\
\leq \|\nabla z_{R_1}(x,0)\|^2 + \lambda^2 \|\nabla E_{R_1}(x,0)\|^2 + M \int_0^t \|(E_{R_1}, \nabla E_{R_1}, z_{R_1}, \nabla z_{R_1})\|^2 dt \\
+ M \lambda^4 \int_0^t (\|\nabla \nabla E_{R_1}\|^2) dt + M \lambda \\
+ M \int_0^t (\|\nabla z_{R_1}\|^2_{H^1} \|E_{R_1}\|^2_{H^1} + \|z_{R_1}\|^2_{H^2} \|\nabla E_{R_1}\|^2) dt \\
+ \|\nabla z_{R_1}\|^2_{H^1} \|E_{R_1}\|^2_{H^1} + M \lambda^4 \int_0^t (\|\nabla E_{R_1}\|^2 \|E_{R_1}\|^2_{H^1} + \|E_{R_1}\|^2_{H^2} \|\nabla \nabla E_{R_1}\|^2) dt \\
+ \|\nabla E_{R_1}\|^2_{H^1} \|E_{R_1}\|^2_{H^1} + M \lambda.
\end{equation}

Step 5. The estimates on \(\|\nabla z_R, \Delta z_R, E_R, \lambda \nabla E_R, \lambda \nabla \nabla E_R\|\|^2\).

From (4.7) + (4.8), by taking \(\delta_2 > 0\) small enough, one obtains

\begin{equation}
\|\nabla z_R\|^2 + C_0 \|E_R\|^2 + \lambda^2 \|\nabla E_R\|^2 \\
\leq - \frac{d}{dt} \|z_R\|^2 + \lambda^2 \|E_R\|^2 + M\|z_R\|^2 + M\|z_R\|_{H^2} \|E_R\|^2.
\end{equation}
\[ + M\lambda^4(\|E_R\|_{H^1} \|\text{div} E_R\|) + M\lambda \]
\[ \leq M\|\langle z_R, z_{Rt}\rangle\|^2 + M\lambda^2\|\langle E_{Rt}, E_{Rt}\rangle\|^2 + M\|z_R\|_{H^2}^2 \|E_R\|^2 \]
\[ + M\lambda^4(\|E_R\|_{H^1} \|\text{div} E_R\|) + M\lambda. \] (4.15)

Multiplying (4.3) by \(-\Delta z_R\), multiplying (4.4) by \(-\Delta \text{div} E_R\), and integrating the resulting equations over \(\Omega\) with respect to \(x\), using the method similar above inequality, one obtains

\[ \|\Delta z_R\|^2 + \lambda^2 \|\nabla \text{div} E_R\|^2 + C_0 \|\text{div} E_R\|^2 \leq M\|\langle z_R, \nabla z_R, \nabla z_{Rt}, E_R\rangle\|^2 + M\lambda^2\|\langle \text{div} E_R, \text{div} E_{Rt}\rangle\|^2 + M\|\nabla z_R\|_{H^1}^2 \|E_R\|_{H^1}^2 \]
\[ + M\|z_R\|_{H^2}^2 \|\text{div} E_R\|^2 + M\lambda^4 \|\text{div} E_R\|^2 \|E_R\|_{H^2}^2 + M\lambda^4 \|\nabla \text{div} E_R\|_{H^1}^2 + M\lambda. \] (4.16)

Step 6. Under the assumption of the Main Theorem, by the standard elliptic regularity estimates, one has

\[ \|z_R\|_{H^2}^2 \leq M(\|z_R\|_{L^2}^2 + \|\Delta z_R\|_{L^2}^2), \] (4.17)
\[ \|z_{Rt}\|_{H^2}^2 \leq M(\|z_{Rt}\|_{L^2}^2 + \|\Delta z_{Rt}\|_{L^2}^2), \] (4.18)
\[ \|E_{Rt}\|_{H^2}^2 \leq M(\|E_{Rt}\|_{L^2}^2 + \|\text{div} E_{Rt}\|_{L^2}^2), \quad s = 1, 2, \] (4.19)
\[ \|E_{Rt}\|_{H^2}^2 \leq M(\|E_{Rt}\|_{L^2}^2 + \|\text{div} E_{Rt}\|_{L^2}^2), \quad s = 1, 2. \] (4.20)

Introduce the following \(\lambda\)-weighted Liapunov-type functional for the remainder terms:

\[ \Gamma^\lambda(t) = \|\langle z_R, \nabla z_R, \Delta z_R, z_{Rt}, \nabla z_{Rt}\rangle\|_{L^2(\Omega)}^2 + \lambda^2 \|\langle E_{Rt}, \text{div} E_R, \nabla \text{div} E_R, E_{Rt}, \text{div} E_{Rt}\rangle\|_{L^2(\Omega)}^2 \]
\[ + \|\langle E_R, \text{div} E_R\rangle\|_{L^2(\Omega)}^2, \]
\[ G^\lambda(t) = \|\langle \Delta z_{Rt}, E_{Rt}, \text{div} E_{Rt}\rangle\|_{L^2(\Omega)}^2 + \lambda^2 \|\nabla \text{div} E_{Rt}\|_{L^2(\Omega)}^2. \]

From (4.9)+\delta_3(4.12)+\delta_4(4.13)+\delta_5(4.14)+\delta_6(4.15)+\delta_7(4.16), by taking \(\delta_j, (j = 3, \cdots, 7)\), one obtains a new inequality. By inserting the standard elliptical regularity estimates (4.17)-(4.20) into the resulting inequality, one gets

\[ \Gamma^\lambda(t) + \int_0^t G^\lambda(s) \, ds \leq M\Gamma^\lambda(t = 0) + M \int_0^t (\Gamma^\lambda(s) + (\Gamma^\lambda(s))^2) \, ds \]
\[ + M(\Gamma^\lambda(t))^2 + M \int_0^t \Gamma^\lambda(s) G^\lambda(s) \, ds + M\lambda, \quad t \geq 0, \] (4.21)

where

\[ \Gamma^\lambda(t = 0) = \|\langle z_R, z_{Rt}, \nabla z_{Rt}\rangle(x, 0)\|_{L^2(\Omega)}^2 + \lambda^2 \|\langle E_{Rt}, E_{Rt}, \text{div} E_{Rt}\rangle(x, 0)\|_{L^2(\Omega)}^2. \]

At last, based on the assumption (3.10)-(3.11) and the error equations (4.3)-(3.3), one can prove that there exists positive constants \(\bar{M}\) and \(\alpha\) such that \(\bar{\Gamma}^\lambda(t = 0) \leq \bar{M} \lambda^\alpha\). With \(\alpha = 1\), one gets (3.12). □
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