Mixed Finite Element Methods for Fourth Order Elliptic Optimal Control Problems

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Abstract. In this paper, a priori error estimates are derived for the mixed finite element discretization of optimal control problems governed by fourth order elliptic partial differential equations. The state and co-state are discretized by Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. The error estimates derived for the state variable as well as those for the control variable seem to be new. We illustrate with a numerical example to confirm our theoretical results.

AMS subject classifications: 65N15, 65N30

Key words: fourth order mixed finite element methods, optimal control problems, $L^2$ projection, a priori error estimates.

1. Introduction

During the last decade the discretization of optimal control problems involving second-order elliptic partial differential equations (introduced in [Lions (1971); Tröltzsch (2010)]) has a number of applications in mathematical and physical problems. For instance, heat conduction, diffusion, electromagnetic waves, fluid flows, freezing processes, and many other physical phenomena can be modeled by partial differential equations. As far as numerical approximation of control problems is concerned, finite element methods (FEMs) play a vital role since these methods have certain advantages over finite difference methods. In many optimal control problems, the objective functional contains not only the primal state variable but also its gradient. The advantage of mixed element methods is that the approximations to $u$ and the flux $p$ can be obtained simultaneously. There are several results available in the literature in which FEMs are used for the numerical approximation of optimal control problems (see [10–12, 14]). Recently there appeared some new progresses on the numerical solution of optimal control problems; this include, the work of Hinze in [17] in which he introduced a new variational discretization approach for linear-quadratic optimal control problems and...
obtained improved convergence order for optimal control in [Hinze (2005)], where the control set is not discretized explicitly. The problems described by bi-harmonic equations arise from fluid mechanics and solid mechanics such as bending of elastic plates. From an application point of view, the interest in higher order elliptic equations as constraints in optimization problems is two fold. First, of course, is the use of plate models as constraints in optimization. Further applications occur in fluid mechanics in the context of the stream function formulation of the Navier-Stokes equations. Second, mixed formulations of fourth order problems have a close connection to necessary conditions for optimization problems governed by second order partial differential equations.

The choice of a mixed discretization of the biharmonic problem is thus, on one hand, motivated by the connection to the bi-level optimization case and on the other by the possibility to approximate this using $H^1$-conforming finite elements instead of the more expensive $H^2$-conforming elements. There has been much research about mixed finite element methods for the 4th order PDEs, for example, Ciarlet-Raviart elements, Herrmann-Miyoshi elements, Hellan-Herrmann-Johnson elements found in (see [4, 9, 19, 27, 29]). In [3], the author has presented estimation of the control error in discretization PDE-constrained optimization. Further the author has estimated the error in the control variable measured in a natural norm. In [23], the author has established vorticity superconvergence of a finite element method for the biharmonic equation by Ciarlet-Raviart’s scheme under the biquadratic uniform rectangular mesh. In [21], the author has developed mixed finite element methods for the fourth-order nonlinear elliptic problem. Optimal $L^2$ error estimates are proved by using a special interpolation operator on the standard tensor-product mixed finite element methods of order $k \geq 1$. In [20], the author has studied a mixed finite element method for the optimal boundary control problem governed by the bi-harmonic equation. The system of optimality equations consisting of state and costate functions is derived. Further for optimality equation based on the system, a gradient-type optimization method is used as a mixed finite approximation for solving the optimal boundary control problem.

Recently in [8], the author has studied a priori error estimates of Ciarlet-Raviart mixed finite element methods for fourth order elliptic control problems with the first biharmonic equation. The optimality conditions consisting of the state and the costate equations are derived. In [18], the author has developed priori error estimates for the finite element discretization of optimal distributed control problems governed by the biharmonic operator. The state equation is discretized in primal mixed form using continuous piecewise biquadratic finite elements, while piecewise constant approximations are used for the control. Further error estimates are derived for the state variable as well as those for the control which are order-optimal on general unstructured meshes. However, on uniform meshes, not all error estimates are optimal due to the low-order control approximation. In [7], the author has developed posteriori error estimates for nonconforming finite element methods for fourth-order problems on rectangles. In [26], the author has presented sufficient conditions for the existence of at least one nontrivial weak solution to a fourth order elliptic problem with a $p(x)$-biharmonic operator and the Navier boundary conditions. In [2, 13, 15, 24, 25], the
authors have established error estimates for linear elliptic optimal control problems by mixed finite element methods with some primary works on a priori, superconvergence and a posteriori error estimates.

The elliptic optimal control problems have many physical applications, for example, laser surface hardening, optimal stationary heating and many other physical processes employing elliptic partial differential equations for their description. In this paper, we study error estimates of mixed finite element methods for optimal control problems governed by elliptic fourth order partial differential equations

\[
\begin{align*}
\minimize_{u \in U_{ad} \subseteq L^2(\Omega)} & \left\{ \frac{1}{2} \left( \| \Delta y \|^2 + \| \nabla y \|^2 + \| y - y_d \|^2 + v \| u \|^2 \right) \right\}, \\
\Delta^2 y &= f + u, \quad x \in \Omega, \\
y &= \Delta y = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with Lipschitz continuous boundary on \( \partial \Omega \). \( U_{ad} \) denotes the admissible set of the control variable defined by

\[
U_{ad} = \left\{ u \in L^2(\Omega) : a(x) \leq u(x) \leq b(x) \text{ a.e in } \Omega \right\}.
\]

Moreover \( f \) and \( y_d \) are given functions and \( v \) is a fixed positive number. We assume that \( \Omega \) is a convex polygon domain.

We have arranged the contents of this paper in the following manner: The remaining part of this section recalls some basic notations of the Sobolev spaces. In Section 3, we formulate fourth order mixed finite element approximations for optimal control problems. In Section 4, preliminaries and estimates of intermediate error are stated. Then we study a priori error estimates for fourth order mixed finite element methods for elliptic control problems with pointwise control constraints, it is proved that these approximations have convergence order \( O(h^{k+1}) \) in Section 5. Finally we illustrate with a numerical example to confirm our theoretical results in Section 6.

\section{2. Notations and Preliminaries}

The following notations will be used throughout the article:

\begin{align*}
\mathcal{T}_h &= \text{a regular simplicial triangulations of } \Omega \\
T &= \text{a triangle of } \mathcal{T}_h \\
\rho(T) &= \text{diameter of the set } T \\
\sigma(T) &= \text{diameter of the largest ball contained in } T \\
h &= \max \{ \rho(T) : T \in \mathcal{T}_h \}
\end{align*}

Furthermore, \([1]\) for \( 1 \leq p \leq \infty \) and \( m \) any nonnegative integer, we consider three vector spaces on which \( \| \cdot \|_{m,p} \) is a norm:
(a) $H^{m,p}(\Omega) \equiv \text{the completion of } \{ \phi \in C^m(\Omega) : \| \phi \|_{m,p} < \infty \} \text{ with respect to the norm } \| \cdot \|_{m,p}$.

(b) $W^{m,p}(\Omega) \equiv \{ \phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m \}$, where $D^\alpha \phi$ is the weak partial derivative of $\phi$, then Sobolev norm is given by

$$||\phi||_{m,p} = \left( \sum_{0 \leq |\alpha| \leq m} ||D^\alpha \phi||_{L^p(\Omega)}^p \right)^{1/p}, \quad ||\phi||_{m,p} = \max_{0 \leq |\alpha| \leq m} ||D^\alpha \phi||_{\infty},$$

and the semi-norm $| \cdot |_{m,p}$ is given by

$$|\phi|_{m,p} = \sum_{|\alpha|=m} ||D^\alpha \phi||_{L^p(\Omega)}.$$

We set $W^{m,p}_0(\Omega) = \{ \phi \in W^{m,p}(\Omega) : \phi \big|_{\partial\Omega} = 0 \}$. For any $m$, we have the obvious chain of imbeddings

$$W^{m,p}_0(\Omega) \rightarrow W^{m,p}(\Omega) \rightarrow L^p(\Omega).$$

For $p = 2$, we denote $H^{m,p}(\Omega) = W^{m,2}(\Omega)$, $H^{m,p}_0(\Omega) = W^{m,2}_0(\Omega)$, $\| \cdot \|_m = ||| \cdot ||_m,2$, and $\| \cdot \| = ||| \cdot ||_0,2$. In addition $C$ denotes a general positive constant independent of $h$, where $h$ is the spatial mesh-size for the control and state discretization.

**Definition 2.1.** If $\Omega \subset \mathbb{R}^2$ is open and measurable, then

$$V = H(\text{div}; \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^2, \text{div} \mathbf{v} \in L^2(\Omega) \right\}.$$ 

The Hilbert space $V$ is equipped with the following norm given by

$$\| \mathbf{v} \|_{H(\text{div}; \Omega)} = \left( \| \mathbf{v} \|^2 + \| \text{div} \mathbf{v} \|^2 \right)^{1/2} \quad \text{and} \quad W = L^2(\Omega).$$

The domain $\Omega$ is said to be $H^{s+2}$-regular if the Dirichlet problem

$$-\Delta \phi = \psi \quad \text{in } \Omega, \quad \phi \big|_{\partial \Omega} = 0$$

(2.1)

is uniquely solvable for $\psi \in L^2$ and if

$$\| \phi \|_{s+2} \leq C\| \psi \|_s \quad \text{for all } \psi \in H^s(\Omega).$$

The present work introduces mixed finite element methods for fourth order elliptic optimal control problems (1.1)-(1.2). The state and adjoint state are discretized by Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. The convergence behavior of discretized problems is studied.
3. Fourth order mixed methods

In this section, we assume that the domain $\Omega$ is $H^{s+2}$-regular for $k = 1$ and that $f \in H^k(\Omega)$ and $y_d \in H^k(\Omega)$.

Let $\tilde{p} = -\nabla \tilde{y}$ and $\tilde{y} = -\Delta y$. Then mixed finite element methods of fourth order optimal control problem is equivalent to (1.1). Then we have

$$\begin{align*}
\min_{u \in U_{ad} \subset L^2(\Omega)} \left\{ \frac{1}{2} \left( \| \tilde{y} \|^2 + \| \tilde{p} \|^2 + \| y - y_d \|^2 + v \| u \|^2 \right) \right\},
\tilde{p} = -\nabla \tilde{y}, \quad x \in \Omega, \\
\tilde{y} \in H^2(\Omega), \quad x \in \Omega, \\
p - \nabla y, \quad x \in \Omega, \\
\div v = f + u, \quad x \in \Omega, \\
y = \tilde{y} = 0, \quad x \in \partial \Omega.
\end{align*}$$

(3.1)

To consider the mixed finite element approximation of general convex fourth order optimal control problems, we need a weak formulation for the state equation (1.1). We recast (1.1) in the following weak form: find $(p, y, \tilde{p}, \tilde{y}, u) \in (V \times W)^2 \times U_{ad}$ such that

$$\begin{align*}
\min_{u \in U_{ad}} \left\{ \frac{1}{2} \left( \| \tilde{y} \|^2 + \| \tilde{p} \|^2 + \| y - y_d \|^2 + v \| u \|^2 \right) \right\},
(y, \div v) = 0, \quad \forall v \in V, \\
(\div \tilde{v}, w) = (\tilde{y}, w), \quad \forall w \in W, \\
(y, \tilde{p} - (\tilde{y}, \div v) = 0, \quad \forall v \in V, \\
(\div \tilde{v}, w) = (\tilde{y}, w), \quad \forall w \in W, \\
\tilde{q} - z, \div v = 0, \quad \forall v \in V, \\
(\div \tilde{v}, w) = (\tilde{y}, w) + (\tilde{y}, w), \quad \forall w \in W, \\
(q, v) - (\tilde{q}, \div v) = 0, \quad \forall v \in V, \\
(\div q, w) = (y-y_d, w), \quad \forall w \in W, \\
\div \tilde{v}, w) = (y - y_d, w), \quad \forall w \in W, \\
u \in U_{ad}.
\end{align*}$$

(3.2a) (3.2b) (3.2c) (3.2d) (3.2e)

It is well known that the optimal control problem (3.2) has a unique solution $(p, y, \tilde{p}, \tilde{y}, u) \in (V \times W)^2 \times U_{ad}$ and that $(p, y, \tilde{p}, \tilde{y}, u)$ is the solution if and only if there is a costate $(q, z, \tilde{q}, \tilde{z}) \in (V \times W)^2$ such that $(p, y, \tilde{p}, \tilde{y}, q, z, \tilde{q}, \tilde{z}, u)$ satisfies the following optimality conditions [22]:

$$\begin{align*}
(y, \div v) = 0, \quad \forall v \in V, \\
(\div \tilde{v}, w) = (\tilde{y}, w), \quad \forall w \in W, \\
(y, \tilde{p} - (\tilde{y}, \div v) = 0, \quad \forall v \in V, \\
(\div \tilde{v}, w) = (f + u, w), \quad \forall w \in W, \\
\tilde{q} - z, \div v = 0, \quad \forall v \in V, \\
(q, v) - (\tilde{q}, \div v) = 0, \quad \forall v \in V, \\
(\div q, w) = (y-y_d, w), \quad \forall w \in W, \\
(vu + z, \tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad}.
\end{align*}$$

(3.3a) (3.3b) (3.3c) (3.3d) (3.3e) (3.3f) (3.3g) (3.3h) (3.3i)
In order to express the control variable in a concise form, we introduce the projection operator

$$\Pi_{[a(x), b(x)]}(f(x)) = \max \{ a(x), \min(b(x), f(x)) \} \quad \text{for every } x \in \Omega.$$  \hfill (3.4)

From the above optimality condition (3.3i) and Remark 3 in [6], we have

$$u(x) = \Pi_{[a(x), b(x)]}\left\{ -\frac{1}{v} z(x) \right\},$$  \hfill (3.5)

where the adjoint state $z(x)$ is the solution to problem (3.3e)-(3.3h). Therefore, with

$$\mu(x) = -(v^{-1} z(x) + u) = -v^{-1} J'(u),$$

we have

$$u(x) = \begin{cases} u_a(x) & \text{if } -v^{-1} z(x) < a(x), \\ -v^{-1} z(x) & \text{if } -v^{-1} z(x) \in [a(x), b(x)], \\ u_b(x) & \text{if } -v^{-1} z(x) > b(x). \end{cases}$$

Let $T_h$ denote a regular triangulation of the domain $\Omega$. Let $V_h \times W_h \subset V \times W$ denote the order of Raviart-Thomas [28] mixed finite element spaces. $P_k$ indicates the space of polynomials of degree at most $k$, where $x = (x_1, x_2)$, with

$$\forall T \in T_h, \quad V(T) = \left\{ P_k^2(T) + x \cdot P_k(T) \right\}, \quad W(T) = P_k(T).$$  \hfill (3.6)

We define the finite dimensional spaces as follows:

$$V_h := \left\{ v_h \in V : \forall T \in T_h, v_h \big| T \in V(T) \right\},$$  \hfill (3.7a)

$$W_h := \left\{ w_h \in W : \forall T \in T_h, w_h \big| T \in W(T) \right\}. $$  \hfill (3.7b)
Associated with $T_h \subset T$ is another finite dimensional subspace of control variable $U^h_{ad}$ \subset \text{U}_{ad}$ given by

$$U^h_{ad} := \{ u_h \in \text{U}_{ad} : \forall T \in T_h, \ u_h|_T = \text{constant} \}. \quad (3.8)$$

Then the mixed finite element discretization of (3.2) is as follows: find $(p, y, \tilde{p}, \tilde{y}, u) \in (\text{V} \times \text{W})^2 \times \text{U}_{ad}$ such that

$$\minimize_{u_h \in U^h_{ad}} \left\{ \frac{1}{2} \left( \| \tilde{y}_h \|^2 + \| \tilde{p}_h \|^2 + \| y_h - y_d \|^2 + v \| u_h \|^2 \right) \right\}, \quad (3.9a)$$

$$(\tilde{p}_h, v_h) - (y_h, \text{div} v_h) = 0, \quad \forall v_h \in \text{V}_h, \quad (3.9b)$$

$$(\text{div} \tilde{p}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in \text{W}_h, \quad (3.9c)$$

$$(p_h, v_h) - (\tilde{y}_h, \text{div} v_h) = 0, \quad \forall v_h \in \text{V}_h, \quad (3.9d)$$

$$(\text{div} p_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in \text{W}_h. \quad (3.9e)$$

It is well known that the optimal control problem (3.9) again has a unique solution $(p_h, y_h, \tilde{p}_h, \tilde{y}_h, u_h) \in (\text{V}_h \times \text{W}_h)^2 \times U^h_{ad}$ and that $(p_h, y_h, \tilde{p}_h, \tilde{y}_h, u_h)$ is the solution if and only if there is a costate $(q_h, z_h, \bar{q}_h, \bar{z}_h) \in (\text{V}_h \times \text{W}_h)^2$ such that $(p_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, z_h, \bar{q}_h, \bar{z}_h, u_h)$ satisfies the following optimality conditions:

$$(\tilde{p}_h, v_h) - (y_h, \text{div} v_h) = 0, \quad \forall v_h \in \text{V}_h, \quad (3.10a)$$

$$(\text{div} \tilde{p}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in \text{W}_h, \quad (3.10b)$$

$$(p_h, v_h) - (\tilde{y}_h, \text{div} v_h) = 0, \quad \forall v_h \in \text{V}_h, \quad (3.10c)$$

$$(\text{div} p_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in \text{W}_h, \quad (3.10d)$$

$$(\bar{q}_h, v_h) - (z_h, \text{div} v_h) = 0, \quad \forall v_h \in \text{V}_h, \quad (3.10e)$$

$$(\text{div} \bar{q}_h, w_h) = (\bar{z}_h, w_h) + (\tilde{y}_h, w_h), \quad \forall w_h \in \text{W}_h, \quad (3.10f)$$

$$(q_h, v_h) - (\bar{z}_h, \text{div} v_h) = -(\tilde{p}_h, v_h), \quad \forall v_h \in \text{V}_h, \quad (3.10g)$$

$$(\text{div} q_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in \text{W}_h, \quad (3.10h)$$

$$(v u_h + z_h, \bar{u}_h - u_h) \geq 0, \quad \forall \bar{u}_h \in \text{U}_{ad}. \quad (3.10i)$$

4. Intermediate error of control problems

In this section, we use some intermediate variables. Now we define the standard $L^2(\Omega)$-orthogonal projection [16] $P_h : \text{W} \rightarrow \text{W}_h$ which satisfies, for any $\phi \in \text{W}$,

$$(\phi - P_h \phi, w_h) = 0, \quad \forall w_h \in \text{W}_h. \quad (4.1)$$

Next we recall the Fortin projection [5] $\Pi_h : \text{V} \rightarrow \text{V}_h$ which satisfies, for any $q \in \text{V}$,

$$(\text{div}(q - \Pi_h q), w_h) = 0, \quad \forall w_h \in \text{W}_h. \quad (4.2)$$

Then the following diagram commutes:
that is, $\text{div} \circ \Pi_h = P_h \circ \text{div} : V \rightarrow W_h$ and $\text{div}(I - \Pi_h)V \perp W_h$ with the following approximation properties:

\[
\begin{align*}
\|q - \Pi_h q\|_{0,\rho,\Omega} & \leq C h^r \|q\|_{r,\rho,\Omega}, \quad \text{for } \frac{1}{\rho} < r \leq k + 1, q \in (W^{1,r}(\Omega))^2, \quad (4.3) \\
\|\phi - P_h \phi\|_{0,\rho,\Omega} & \leq C h^t \|\phi\|_{t,\rho,\Omega}, \quad \text{for } 0 \leq t \leq k + 1, \phi \in W^{1,\rho}(\Omega), \quad (4.4) \\
\|\text{div}(q - \Pi_h q)\|_t & \leq C h^{t+r} \|\text{div}q\|_r, \quad \text{for } t \leq k + 1, \text{div}q \in W^{1,r}(\Omega), \quad (4.5) \\
\|\phi - P_h \phi\|_{-t,\rho,\Omega} & \leq C h^{t+r} \|\phi\|_{r,\rho,\Omega}, \quad \text{for } t \leq k + 1, \phi \in H^1(\Omega), \quad (4.6)
\end{align*}
\]

where $\| \cdot \|_{r,\rho,\Omega}$ denotes the norm of the usual Sobolev space $W^{r,\rho}(\Omega)$ for $1 \leq \rho \leq \infty$ and $r \geq 0$.

Furthermore we also define the standard $L^2(\Omega)$-orthogonal projection $Q_h : U \rightarrow U_h$ which satisfies: for any $\tilde{u} \in U$,

\[
(\tilde{u} - Q_h \tilde{u}, u_h) = 0, \quad \forall u_h \in U_h. \quad (4.7)
\]

Then $Q_h \tilde{u}$ is in fact just element average of $\tilde{u}$. Similar to (4.6), we have the approximation property:

\[
\|\tilde{u} - Q_h \tilde{u}\|_{r,U} \leq C h^r |\tilde{u}|_{1,r,U}, \quad \forall \tilde{u} \in W^{1,r}(\Omega). \quad (4.8)
\]

For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(p(\tilde{u}), y(\tilde{u}), \tilde{p}(\tilde{u}), \tilde{y}(\tilde{u}), q(\tilde{u}), z(\tilde{u}), \tilde{q}(\tilde{u}), \tilde{z}(\tilde{u}))$ associated with $\tilde{u}$ which satisfies

\[
\begin{align*}
(p(\tilde{u}), v) - (y(\tilde{u}), \text{div}v) & = 0, \quad \forall v \in V, \quad (4.9a) \\
(\text{div}\tilde{p}(\tilde{u}), w) & = (\tilde{y}(\tilde{u}), w), \quad \forall w \in W, \quad (4.9b) \\
(p(\tilde{u}), v) - (\tilde{y}(\tilde{u}), \text{div}v) & = 0, \quad \forall v \in V, \quad (4.9c) \\
(\text{div}\tilde{p}(\tilde{u}), w) & = (f + \tilde{u}, w), \quad \forall w \in W, \quad (4.9d) \\
(\tilde{q}(\tilde{u}), v) - (z(\tilde{u}), \text{div}v) & = 0, \quad \forall v \in V, \quad (4.9e) \\
(\text{div}\tilde{q}(\tilde{u}), w) & = (\tilde{z}(\tilde{u}), w) + (\tilde{y}(\tilde{u}), w), \quad \forall w \in W, \quad (4.9f) \\
(q(\tilde{u}), v) - (\tilde{z}(\tilde{u}), \text{div}v) & = - (\tilde{p}(\tilde{u}), v), \quad \forall v \in V, \quad (4.9g) \\
(\text{div}\tilde{q}(\tilde{u}), w) & = (y(\tilde{u}) - y_d, w), \quad \forall w \in W. \quad (4.9h)
\end{align*}
\]
Then we define the discrete state solution \((p(\tilde{u}_h), y(\tilde{u}_h), \tilde{p}(\tilde{u}_h), \tilde{y}(\tilde{u}_h), q(\tilde{u}_h), z(\tilde{u}_h), \tilde{q}(\tilde{u}_h), \tilde{z}(\tilde{u}_h)) \in (V_h \times W_h)^4\) associated with \(\tilde{u}\) which satisfies
\[
(p(\tilde{u}_h), \nu_h) - (y(\tilde{u}_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.10a)
\]
\[
(\text{div} p(\tilde{u}_h), w_h) = (\tilde{y}(\tilde{u}_h), w_h), \quad \forall w_h \in W_h, \quad (4.10b)
\]
\[
(p(\tilde{u}_h), \nu_h) - (\tilde{y}(\tilde{u}_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.10c)
\]
\[
(\text{div} p(\tilde{u}_h), w_h) = (f + \tilde{u}_h, w_h), \quad \forall w_h \in W_h, \quad (4.10d)
\]
\[
(\tilde{q}(\tilde{u}_h), \nu_h) - (\tilde{z}(\tilde{u}_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.10e)
\]
\[
(\text{div} q(\tilde{u}_h), w_h) = (\tilde{z}(\tilde{u}_h), w_h) + (\tilde{y}(\tilde{u}_h), w_h), \quad \forall w_h \in W_h, \quad (4.10f)
\]
\[
(\tilde{q}(\tilde{u}_h), \nu_h) - (\tilde{z}(\tilde{u}_h), \text{div} \nu_h) = - (\tilde{p}(\tilde{u}_h), \nu_h), \quad \forall \nu_h \in V_h, \quad (4.10g)
\]
\[
(\text{div} q(\tilde{u}_h), w_h) = (y(\tilde{u}_h) - y_d, w_h), \quad \forall w_h \in W_h. \quad (4.10h)
\]

With these definitions, the exact state and co-state solutions and their approximation can be written as
\[
(p, y, \tilde{p}, \tilde{y}, q, z, \tilde{q}, \tilde{z}) = (p(u), y(u), \tilde{p}(u), \tilde{y}(u), q(u), z(u), \tilde{q}(u), \tilde{z}(u)), \quad (4.11)
\]
\[
(p_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, z_h, \tilde{q}_h, \tilde{z}_h) = (p_h(u_h), y_h(u_h), \tilde{p}_h(u_h), \tilde{y}_h(u_h), q_h(u_h), z_h(u_h), \tilde{q}_h(u_h), \tilde{z}_h(u_h)). \quad (4.12)
\]

**Lemma 4.1.** Let \((p_h, \tilde{y}_h, p_h, y_h, \tilde{q}_h, q_h, z_h, u_h)\) be an optimal solution of (3.3). Then \(u_h\) is given by
\[
u_h(x) = \Pi_{[a(x), b(x)]} \left\{ - \frac{1}{u} \frac{\partial}{\partial x} \right\}. \quad (4.13)
\]

In the following, we further assume that \(\tilde{p}, \tilde{y}, p, y, u\) are locally Lipschitz continuous and that there is a constant \(C > 0\) such that
\[
(J'(u) - J'(v), u - v) \geq C \|u - v\|^2_{U_{ad}}, \quad \forall u, v \in U_{ad}. \quad (4.14)
\]

Now, if we choose \(\tilde{u}_h = u_h\) in (4.10), we obtain the intermediate solution \((p(u_h), y(u_h), \tilde{p}(u_h), \tilde{y}(u_h), q(u_h), z(u_h), \tilde{q}(u_h), \tilde{z}(u_h)) \in (V_h \times W_h)^4\) satisfying the following equations:
\[
(p(u_h), \nu_h) - (y(u_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.15a)
\]
\[
(\text{div} p(u_h), w_h) = (\tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \quad (4.15b)
\]
\[
(p(u_h), \nu_h) - (\tilde{y}(u_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.15c)
\]
\[
(\text{div} p(u_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, \quad (4.15d)
\]
\[
(q(u_h), \nu_h) - (\tilde{z}(u_h), \text{div} \nu_h) = 0, \quad \forall \nu_h \in V_h, \quad (4.15e)
\]
\[
(\text{div} q(u_h), w_h) = (\tilde{z}(u_h), w_h) + (\tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \quad (4.15f)
\]
\[
(q(u_h), \nu_h) - (\tilde{z}(u_h), \text{div} \nu_h) = - (\tilde{p}(u_h), \nu_h), \quad \forall \nu_h \in V_h, \quad (4.15g)
\]
\[
(\text{div} q(u_h), w_h) = (y(u_h) - y_d, w_h), \quad \forall w_h \in W_h. \quad (4.15h)
\]

where \((p(u_h), y(u_h), \tilde{p}(u_h), \tilde{y}(u_h), q(u_h), z(u_h), \tilde{q}(u_h), \tilde{z}(u_h)) \in (V_h \times W_h)^4\) is a solution for (4.9). We establish the following lemma.
Lemma 4.2. Let \((p_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, z_h, \tilde{q}_h, \tilde{z}_h) \in (V_h \times W_h)^4\) be the solution of (3.10) and \((p(u_h), y(u_h), \tilde{p}(u_h), \tilde{y}(u_h), q(u_h), z(u_h), \tilde{q}(u_h), \tilde{z}(u_h)) \in (V_h \times W_h)^4\) be the solution of (4.15) respectively. Assume that \(y, \tilde{y}, z, \tilde{z} \in H^{k+1}(\Omega)\), \(p, \tilde{p}, q, \tilde{q} \in (H^{k+1}(\Omega))^2\) and \(\text{div} p, \text{div} \tilde{p}, \text{div} q, \text{div} \tilde{q} \in H^{k+1}(\Omega)\). Then we have

\[
\begin{align*}
\|y_h - y(u_h)\| + \|p_h - p(u_h)\|_{\text{div}} &\leq Ch^{k+1}, \quad (4.16) \\
\|\tilde{y}_h - \tilde{y}(u_h)\| + \|\tilde{p}_h - \tilde{p}(u_h)\|_{\text{div}} &\leq Ch^{k+1}, \quad (4.17) \\
\|z_h - z(u_h)\| + \|q_h - q(u_h)\|_{\text{div}} &\leq Ch^{k+1}, \quad (4.18) \\
\|\tilde{z}_h - \tilde{z}(u_h)\| + \|\tilde{q}_h - \tilde{q}(u_h)\|_{\text{div}} &\leq Ch^{k+1}. \quad (4.19)
\end{align*}
\]

Proof. From Equations (3.10) and (4.15), set some intermediate error

\[
\begin{align*}
\varepsilon_1 &= p_h - p(u_h), \quad e_1 = y_h - y(u_h), \\
\varepsilon_2 &= \tilde{p}_h - \tilde{p}(u_h), \quad e_2 = \tilde{y}_h - \tilde{y}(u_h), \\
\varepsilon_3 &= q_h - q(u_h), \quad e_3 = z_h - z(u_h), \\
\varepsilon_4 &= \tilde{q}_h - \tilde{q}(u_h), \quad e_4 = \tilde{z}_h - \tilde{z}(u_h).
\end{align*}
\]

Then we obtain the following error equations

\[
\begin{align*}
(p_h - p(u_h), v_h) - (y_h - y(u_h), \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.20a) \\
(\text{div}(p_h - p(u_h), w_h) = (\tilde{y}_h - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \quad (4.20b) \\
(p_h - p(u_h), v_h) - (\tilde{y}_h - \tilde{y}(u_h), \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.20c) \\
(\text{div}(p_h - p(u_h), w_h) = 0, \quad \forall w_h \in W_h, \quad (4.20d) \\
(q_h - q(u_h), v_h) - (z_h - z(u_h), \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.20e) \\
(\text{div}(q_h - q(u_h), w_h) = (\tilde{z}_h - \tilde{z}(u_h), w_h) + (\tilde{y}_h - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \quad (4.20f) \\
(q_h - q(u_h), v_h) - (\tilde{z}_h - \tilde{z}(u_h), \text{div} v_h) = - (\tilde{p}_h - \tilde{p}(u_h), v_h), \quad \forall v_h \in V_h, \quad (4.20g) \\
(\text{div}(q_h - q(u_h), w_h) = (y_h - y(u_h), w_h), \quad \forall w_h \in W_h. \quad (4.20h)
\end{align*}
\]

Then the equations become with intermediate error estimates

\[
\begin{align*}
(\varepsilon_2, v_h) - (e_1, \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.21a) \\
(\text{div}\varepsilon_2, w_h) = (e_2, w_h), \quad \forall w_h \in W_h, \quad (4.21b) \\
(\varepsilon_1, v_h) - (\varepsilon_2, \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.21c) \\
(\text{div}\varepsilon_1, w_h) = 0, \quad \forall w_h \in W_h, \quad (4.21d) \\
(\varepsilon_4, v_h) - (\varepsilon_3, \text{div} v_h) &= 0, \quad \forall v_h \in V_h, \quad (4.21e) \\
(\text{div}\varepsilon_4, w_h) = (e_4, w_h) + (e_2, w_h), \quad \forall w_h \in W_h, \quad (4.21f) \\
(\varepsilon_3, v_h) - (e_4, \text{div} v_h) = -(\varepsilon_2, v_h), \quad \forall v_h \in V_h, \quad (4.21g) \\
(\text{div}\varepsilon_3, w_h) = (e_1, w_h), \quad \forall w_h \in W_h. \quad (4.21h)
\end{align*}
\]
From Theorem 4.1 in [16], it follows from the standard mixed finite element stability estimates (4.3) and (4.6),

\[ \|
\hat{y}_h - \tilde{y}(u_h)\| + \|p_h - p(u_h)\|_{\text{div}} \leq Ch^{k+1}, \] (4.22)
\[ \|y_h - y(u_h)\| + \|
\hat{p}_h - \tilde{p}(u_h)\|_{\text{div}} \leq Ch^{k+1} + C\|y_h - y(u_h)\|, \] (4.23)
\[ \|
\tilde{z}_h - \tilde{z}(u_h)\| + \|q_h - q(u_h)\|_{\text{div}} \leq Ch^{k+1} + C\left(\|y_h - y(u_h)\| + \|\hat{p}_h - \tilde{p}(u_h)\|\right), \] (4.24)
\[ \|z_h - z(u_h)\| + \|\tilde{q}_h - \tilde{q}(u_h)\|_{\text{div}} \leq Ch^{k+1} + C\left(\|z_h - \tilde{z}(u_h)\| + \|\tilde{y}_h - \tilde{y}(u_h)\|\right). \] (4.25)

Thus (4.16)-(4.19) can be proved from (4.22)-(4.25). \qed

Then, we will give the following results for the intermediate solutions which are very important for our following work.

**Lemma 4.3.** Let \((p(u_h), y(u_h), \tilde{p}(u_h), \tilde{y}(u_h), q(u_h), z(u_h), \tilde{q}(u_h), \tilde{z}(u_h)) \in (V_h \times W_h)^d\) be the solution of (4.10) with \(\tilde{u}_h = u_h\) and \((p, y, \tilde{p}, \tilde{y}, q, z, \tilde{q}, \tilde{z}) \in (V_h \times W_h)^d\) be the solution of (3.3a)-(3.3h), respectively. Assume that \(y, \tilde{y}, z, \tilde{z} \in H^{k+1}(\Omega), p, \tilde{p}, q, \tilde{q} \in (H^{k+1}(\Omega))^2\) and \(\text{div} p, \text{div} \tilde{p}, \text{div} q, \text{div} \tilde{q} \in H^k(\Omega)\). Then we have

\[ \|P_h y - y(u_h)\| + \|P_h z - z(u_h)\| \leq Ch^{k+2}, \] (4.26)
\[ \|P_h \tilde{y} - \tilde{y}(u_h)\| + \|P_h \tilde{z} - \tilde{z}(u_h)\| \leq Ch^{k+2}. \] (4.27)

**Proof.** Using (4.1), we can rewrite (4.20) as

\[
(p - \tilde{p}(u_h), v_h) - (P_h y - y(u_h), \text{div} v_h) = 0, \quad \forall v_h \in V_h, \tag{4.28a}
\]
\[
(\text{div}(\tilde{p} - \tilde{p}(u_h)), w_h) = (P_h \tilde{y} - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \tag{4.28b}
\]
\[
(p - p(u_h), v_h) - (P_h \tilde{y} - \tilde{y}(u_h), \text{div} v_h) = 0, \quad \forall v_h \in V_h, \tag{4.28c}
\]
\[
(\text{div}(p - p(u_h)), w_h) = 0, \quad \forall w_h \in W_h, \tag{4.28d}
\]
\[
(q - \tilde{q}(u_h), v_h) - (P_h z - z(u_h), \text{div} v_h) = 0, \quad \forall v_h \in V_h, \tag{4.28e}
\]
\[
(\text{div}(q - \tilde{q}(u_h)), w_h) = (P_h \tilde{z} - \tilde{z}(u_h), w_h) + (P_h \tilde{y} - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h, \tag{4.28f}
\]
\[
(q - q(u_h), v_h) - (P_h \tilde{z} - \tilde{z}(u_h), \text{div} v_h) = -(\tilde{p} - \tilde{p}(u_h), v_h), \quad \forall v_h \in V_h, \tag{4.28g}
\]
\[
(\text{div}(q - q(u_h)), w_h) = (P_h y - y(u_h), w_h), \quad \forall w_h \in W_h. \tag{4.28h}
\]

For sake of simplicity, we now denote

\[ e = P_h \tilde{z} - \tilde{z}(u_h). \]

Since

\[ \|e\| = \sup_{\psi \in L^2(\Omega), \psi \neq 0} \frac{(e, \psi)}{\|\psi\|}, \] (4.29)
we then need to bound \((e, \psi)\) for \(\psi \in L^2(\Omega)\). Let \(\phi \in H^2(\Omega) \cap H^1_0(\Omega)\) be the solution of (2.1). We can see from (4.2) and (4.28g)

\[
(e, \psi) = (e, -\text{div}(\nabla \phi)) = -(e, \text{div}(\Pi_h(\nabla \phi))) = -(q - q(u_h), \Pi_h(\nabla \phi)) - (\tilde{p} - \tilde{p}(u_h), \Pi_h(\nabla \phi)).
\]

(4.30)

Note that

\[
(\text{div}(q - q(u_h)), \phi) + (q - q(u_h), \nabla \phi) = 0.
\]

(4.31)

Thus, from (4.3)-(4.6), (4.28a), (4.28h), (4.30), (4.31) and Lemma(4.2), we derive that

\[
(e, \psi) = \langle q - q(u_h), \nabla \phi - \Pi_h(\nabla \phi) \rangle + (\text{div}(q - q(u_h)), \phi - P_h \phi) + (P_h y - y(u_h), P_h \phi) - (P_h y - y(u_h), \text{div}(\Pi_h(\nabla \phi))) \leq C(h^{k+2} + \|P_h y - y(u_h)\|)\|\phi\|_2.
\]

(4.32)

Similarly, using the duality argument, we can find that

\[
\|P_h \tilde{y} - \tilde{y}(u_h)\| \leq C h^{k+2},
\]

(4.33)

\[
\|P_h y - y_h\| \leq C h^{k+2} + C \|P_h \tilde{y} - \tilde{y}(u_h)\|,
\]

(4.34)

\[
\|P_h z - z(u_h)\| \leq C h^{k+2} + C \left(\|P_h \tilde{y} - \tilde{y}(u_h)\| + \|P_h \tilde{z} - \tilde{z}(u_h)\| \right).
\]

(4.35)

Thus, by (4.29) and (4.32)-(4.35), we complete the proof. \(\square\)

5. Error estimates of control problems

In this section, we give error estimates for both control variable and the state variable approximations. We assume that \(U_{ad}^h \subset U_{ad} \subset L^2(\Omega)\). Let \((p, y, \tilde{p}, \tilde{y}, q, \tilde{q}, z, \tilde{z}, u) \in (V \times W)^4 \times U_{ad}\) and \((\tilde{p}_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, \tilde{q}_h, z_h, \tilde{z}_h, u_h) \in (V_h \times W_h)^4 \times U_{ad}\) be the solution of (3.3) and (3.10) respectively.

With the intermediate errors, we can decompose the error as follows:

\[
p - \tilde{p}_h := p - p(u_h) + p(u_h) - \tilde{p}_h = \epsilon_1 + \epsilon_1,
\]

\[
\tilde{p} - \tilde{p}_h := \tilde{p} - \tilde{p}(u_h) + \tilde{p}(u_h) - \tilde{p}_h = \epsilon_2 + \epsilon_2,
\]

\[
y - y_h := y - y(u_h) + y(u_h) - y_h = r_1 + \epsilon_1,
\]

\[
\tilde{y} - \tilde{y}_h := \tilde{y} - \tilde{y}(u_h) + \tilde{y}(u_h) - \tilde{y}_h = r_2 + \epsilon_2,
\]

\[
q - q_h := q - q(u_h) + q(u_h) - q_h = \epsilon_3 + \epsilon_3,
\]

\[
\tilde{q} - \tilde{q}_h := \tilde{q} - \tilde{q}(u_h) + \tilde{q}(u_h) - \tilde{q}_h = \epsilon_4 + \epsilon_4,
\]

\[
z - z_h := z - z(u_h) + z(u_h) - z_h = r_3 + \epsilon_3,
\]

\[
\tilde{z} - \tilde{z}_h := \tilde{z} - \tilde{z}(u_h) + \tilde{z}(u_h) - \tilde{z}_h = r_4 + \epsilon_4.
\]
From (3.3a)-(3.3h) and (4.15), we derive the following error equations:

\[(\tilde{p} - \tilde{p}(u_h), v) - (y - y(u_h), \text{div}v) = 0, \quad \forall v \in V, \quad (5.1a)\]

\[\text{div}((\tilde{p} - \tilde{p}(u_h)), w) = (y - \tilde{y}(u_h), w), \quad \forall w \in W, \quad (5.1b)\]

\[(p - p(u_h), v) - (\tilde{y} - \tilde{y}(u_h), \text{div}v) = 0, \quad \forall v \in V, \quad (5.1c)\]

\[\text{div}(p - p(u_h), w) = (u - u_h, w), \quad \forall w \in W, \quad (5.1d)\]

\[(\tilde{q} - \tilde{q}(u_h), v) - (z - z(u_h), \text{div}v) = 0, \quad \forall v \in V, \quad (5.1e)\]

\[\text{div}(\tilde{q} - \tilde{q}(u_h), w) = (\tilde{z} - \tilde{z}(u_h), w) + (\tilde{y} - , w), \quad \forall w \in W, \quad (5.1f)\]

\[(q - q(u_h), v) - (\tilde{z} - \tilde{z}(u_h), \text{div}v) = -(\tilde{p} - \tilde{p}(u_h), v), \quad \forall v \in V, \quad (5.1g)\]

\[\text{div}(q - q(u_h), w) = (y - y(u_h), w), \quad \forall w \in W. \quad (5.1h)\]

Then the equations become with intermediate error estimates

\[(e_2, v) - (r_1, \text{div}v) = 0, \quad \forall v \in V, \quad (5.2a)\]

\[\text{div}e_2, w) = (r_2, w), \quad \forall w \in W, \quad (5.2b)\]

\[(e_1, v) - (r_2, \text{div}v) = 0, \quad \forall v \in V, \quad (5.2c)\]

\[\text{div}e_1, w) = (u - u_h, w), \quad \forall w \in W, \quad (5.2d)\]

\[(e_4, v) - (r_3, \text{div}v) = 0, \quad \forall v \in V, \quad (5.2e)\]

\[\text{div}e_4, w) = (r_4, w) + (r_2, w), \quad \forall w \in W, \quad (5.2f)\]

\[(e_3, v) - (r_4, \text{div}v) = -(e_2, v), \quad \forall v \in V, \quad (5.2g)\]

\[\text{div}e_3, w) = (r_1, w), \quad \forall w \in W. \quad (5.2h)\]

From the standard mixed finite element stability estimate of the saddle-point problem (5.2), we derive

\[\|e_1\|_{\text{div}} + \|r_1\| \leq C\|u - u_h\|_{L^2}, \quad (5.3)\]

\[\|e_3\|_{\text{div}} + \|r_3\| \leq C\left(\|p(u_h) - p\| + \|y - y(u_h)\|ight) \leq C\|u - u_h\|_{L^2}, \quad (5.4)\]

\[\|e_2\|_{\text{div}} + \|r_2\| \leq C\|u - u_h\|_{L^2}, \quad (5.5)\]

\[\|e_4\|_{\text{div}} + \|r_4\| \leq C\left(\|q(u_h) - q\| + \|z - z(u_h)\|ight) \leq C\|u - u_h\|_{L^2}. \quad (5.6)\]

Now, we can derive $L^2$ and $L^\infty$-error estimates for the control variable.

**Lemma 5.1.** Let $(p(P_h u), y(P_h u), \tilde{p}(P_h u), \tilde{y}(P_h u), q(P_h u), z(P_h u), \tilde{q}(P_h u), \tilde{z}(P_h u)) \in (V_h \times W_h)^4$ and $(p(u_h), y(u_h), \tilde{p}(u_h), \tilde{y}(u_h), q(u_h), z(u_h), \tilde{q}(u_h), \tilde{z}(u_h)) \in (V_h \times W_h)^4$ be the solution of (4.10) with $\tilde{u}_h = P_h u$ and $\tilde{u}_h = u_h$, respectively. Then, we have

\[\|y(u_h) - y(P_h u)\| + \|p(u_h) - p(P_h u)\| = 0, \quad (5.7)\]

\[\|z(u_h) - z(P_h u)\| + \|q(u_h) - q(P_h u)\| = 0, \quad (5.8)\]

\[\|\tilde{y}(u_h) - \tilde{y}(P_h u)\| + \|\tilde{p}(u_h) - \tilde{p}(P_h u)\| = 0, \quad (5.9)\]

\[\|\tilde{z}(u_h) - \tilde{z}(P_h u)\| + \|\tilde{q}(u_h) - \tilde{q}(P_h u)\| = 0. \quad (5.10)\]
**Lemma 5.2.** Let $u$ be the solution of (3.3) and $u_h$ be the solution of (3.10), respectively. Let all the assumptions of Lemma 4.3 and $u \in W^{1,\infty}$ be valid. Then, we have
\[
\|P_h u - u_h\| \leq Ch^{k+2}.
\] (5.11)

**Theorem 5.1.** Let $(p, y, \tilde{p}, \tilde{y}, q, z, \tilde{q}, \tilde{z})$ and $(p_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, z_h, \tilde{q}_h, \tilde{z}_h)$ be the solution of (3.7) and (3.10), respectively. Let all the assumption of Lemma 4.3 be valid. Assume that $\mathbf{div} p, \mathbf{div} \tilde{p}, \mathbf{div} q, \mathbf{div} \tilde{q} \in W^{k+1,\infty}(\Omega)$. Then we have
\[
\|y - y_h\|_{0,\infty} + \|\mathbf{div}(p - p_h)\|_{0,\infty} \leq Ch^{k+1},
\] (5.12)
\[
\|\tilde{y} - \tilde{y}_h\|_{0,\infty} + \|\mathbf{div}(\tilde{p} - \tilde{p}_h)\|_{0,\infty} \leq Ch^{k+1},
\] (5.13)
\[
\|z - z_h\|_{0,\infty} + \|\mathbf{div}(q - q_h)\|_{0,\infty} \leq Ch^{k+1},
\] (5.14)
\[
\|\tilde{z} - \tilde{z}_h\|_{0,\infty} + \|\mathbf{div}(\tilde{q} - \tilde{q}_h)\|_{0,\infty} \leq Ch^{k+1}.
\] (5.15)

**Proof.** Using (4.2), we rewrite (4.28) as
\[
\mathbf{div}(p - p_h, w_h) = (P_h y - y_h, \mathbf{div} w_h) = 0, \quad \forall w_h \in V_h, \tag{5.16a}
\]
\[
\mathbf{div}(\tilde{p} - \tilde{p}_h, w_h) = (P_h \tilde{y} - \tilde{y}_h, \mathbf{div} w_h), \quad \forall w_h \in W_h, \tag{5.16b}
\]
\[
(p - p_h, w_h) = (P_h \tilde{y} - \tilde{y}_h, w_h), \quad \forall w_h \in V_h, \tag{5.16c}
\]
\[
\mathbf{div}(p - p_h, w_h) = 0, \quad \forall w_h \in W_h, \tag{5.16d}
\]
\[
(q - q_h, w_h) - (P_h \tilde{z} - \tilde{z}_h, \mathbf{div} w_h) = 0, \quad \forall w_h \in V_h, \tag{5.16e}
\]
\[
\mathbf{div}(q - q_h, w_h) = (P_h \tilde{y} - \tilde{y}_h, w_h) + (P_h \tilde{y} - \tilde{y}_h, w_h), \quad \forall w_h \in W_h, \tag{5.16f}
\]
\[
(q - q_h, w_h) - (P_h \tilde{z} - \tilde{z}_h, \mathbf{div} w_h) = (\tilde{p} - \tilde{p}_h, w_h), \quad \forall w_h \in V_h, \tag{5.16g}
\]
\[
\mathbf{div}(q - q_h, w_h) = (P_h y - y_h, w_h), \quad \forall w_h \in W_h. \tag{5.16h}
\]

Similar to Lemma 4.3, we can prove that
\[
\|P_h y - y_h\| + \|P_h z - z_h\| \leq Ch^{k+2}, \tag{5.17}
\]
\[
\|P_h \tilde{y} - \tilde{y}_h\| + \|P_h \tilde{z} - \tilde{z}_h\| \leq Ch^{k+2}. \tag{5.18}
\]

By use of (4.4), (5.17), (5.18), we can prove that
\[
\|y - y_h\|_{0,\infty} + \|z - z_h\|_{0,\infty} \leq Ch^{k+1}, \tag{5.19}
\]
\[
\|\tilde{y} - \tilde{y}_h\|_{0,\infty} + \|\tilde{z} - \tilde{z}_h\|_{0,\infty} \leq Ch^{k+1}. \tag{5.20}
\]

From (5.16b), (5.16d), (5.16f) and (5.16h), it is easy to derive that
\[
\mathbf{div}(\Pi_h p - \tilde{p}_h) = P_h \tilde{y} - \tilde{y}_h, \tag{5.21}
\]
\[
\mathbf{div}(\Pi_h p - \tilde{p}_h) = P_h \tilde{y} - \tilde{y}_h, \tag{5.22}
\]
\[
\mathbf{div}(\Pi_h q - \tilde{q}_h) = P_h \tilde{y} - \tilde{y}_h + P_h \tilde{z} - \tilde{z}_h, \tag{5.23}
\]
\[
\mathbf{div}(\Pi_h q - \tilde{q}_h) = P_h y - y_h. \tag{5.24}
\]

Combining (4.5), (5.19)-(5.24) with that inverse estimate, we complete the proof of Theorem 5.1. \(\square\)
**Theorem 5.2.** Let $u$ and $u_h$ be the solutions of (3.3) and (3.10). Let all the assumption of Lemma 4.3 be valid. Assume that $u$ in $W^{k+1,\infty}(\Omega)$. Then we have

$$
\|u - u_h\|_{0,\infty} \leq Ch^{k+1}.
$$

(5.25)

We used the local Lipschitz continuity (4.14) of the functionals $\tilde{q}, q, \tilde{z}, z$. In the following, we estimate $\|u - u_h\|_{U_{ad}}$ and then obtain the results.

**Theorem 5.3.** Let $(p, y, \tilde{p}, \tilde{y}, q, z, \tilde{q}, \tilde{z}, u) \in (V \times W)^4 \times U_{ad}$ and $(p_h, y_h, \tilde{p}_h, \tilde{y}_h, q_h, z_h, \tilde{q}_h, z_h, u_h) \in (V_h \times W_h)^4 \times U_{ad}$ be the solutions of (3.3) and (3.10) respectively. Then we have

$$
\|u - u_h\| \leq Ch^{k+1},
$$

(5.26)

$$
\|y - y_h\| + \|p - p_h\|_{\text{div}} \leq Ch^{k+1},
$$

(5.27)

$$
\|\tilde{y} - \tilde{y}_h\| + \|\tilde{p} - \tilde{p}_h\|_{\text{div}} \leq Ch^{k+1},
$$

(5.28)

$$
\|z - z_h\| + \|q - q_h\|_{\text{div}} \leq Ch^{k+1},
$$

(5.29)

$$
\|\tilde{z} - \tilde{z}_h\| + \|\tilde{q} - \tilde{q}_h\|_{\text{div}} \leq Ch^{k+1}.
$$

(5.30)

**Proof.** Part I. Using (4.14), note that the projection $\Pi_{[a(x),b(x)]} f(x)$ defined in (3.4) is Lipschitz continuous. From (3.5) and Lemma 4.1, we derive

$$
|u - u_h| = \left| \Pi_{[a(x),b(x)]} \left\{ - \frac{1}{v} \tilde{z}(x) \right\} - \Pi_{[a(x),b(x)]} \left\{ - \frac{1}{v} \tilde{z}_h(x) \right\} \right|.
$$

(5.31)

Hence, combining (5.4) and (5.31), we get

$$
\|u - u_h\| \leq c \|z - z_h\| \leq Ch^{k+1}.
$$

(5.32)

Thus (5.26) is proved.

**Part II.** Using (3.10) and (4.15), we obtain the following error equations:

$$
(p_h - \tilde{p}(u_h), w_h) - (y_h - y(u_h), \text{div}w_h) = 0, \quad \forall w_h \in V_h,
$$

(5.33a)

$$
(\text{div}(\tilde{p}_h - \tilde{p}(u_h), w_h)) = (\tilde{y}_h - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h,
$$

(5.33b)

$$
(p - p(u_h), v_h) - (\tilde{y}_h - \tilde{y}(u_h), \text{div}v_h) = 0, \quad \forall v_h \in V_h,
$$

(5.33c)

$$
(\text{div}(p_h - p(u_h), w_h)) = 0, \quad \forall w_h \in W_h,
$$

(5.33d)

$$
(q_h - \tilde{q}(u_h), w_h) - (z_h - z(u_h), \text{div}w_h) = 0, \quad \forall w_h \in V_h,
$$

(5.33e)

$$
(\text{div}(q_h - \tilde{q}(u_h), w_h)) = (\tilde{z}_h - \tilde{z}(u_h), w_h) + (\tilde{y}_h - \tilde{y}(u_h), w_h), \quad \forall w_h \in W_h,
$$

(5.33f)

$$
(q_h - q(u_h), v_h) - (\tilde{z}_h - \tilde{z}(u_h), \text{div}v_h) = - (\tilde{p}_h - \tilde{p}(u_h), v_h), \quad \forall v_h \in V_h,
$$

(5.33g)

$$
(\text{div}(q_h - q(u_h), w_h)) = (y_h - y(u_h), w_h), \quad \forall w_h \in W_h.
$$

(5.33h)
From the standard mixed finite element stability estimates of the saddle-point problem, we have

\[
\|p_h - p(u_h)\|_{\text{div}} + \|y_h - y(u_h)\| \leq C\|u - u_h\|, \\
\|q_h - q(u_h)\|_{\text{div}} + \|z_h - z(u_h)\| \leq C\|u - u_h\|, \\
\|\tilde{p}_h - \tilde{p}(u_h)\|_{\text{div}} + \|\tilde{y}_h - \tilde{y}(u_h)\| \leq C\|u - u_h\|, \\
\|\tilde{q}_h - \tilde{q}(u_h)\|_{\text{div}} + \|\tilde{z}_h - \tilde{z}(u_h)\| \leq C\|u - u_h\|.
\]

Thus (5.27)-(5.30) is proved from (5.33f)-(5.34) and Lemma 4.2. □

6. Numerical example

We illustrate the theoretical results of the previous section with a numerical example to obtain the error analysis of mixed finite element methods for fourth order elliptic optimal control problems. The control function \(u\) is discretized by piecewise constant function, whereas the state \((y, p, \tilde{p}, \tilde{y})\) and the costate \((z, q, \tilde{q}, \tilde{z})\) are approximated by Raviart-Thomas mixed finite element methods solved numerically by primal-dual active set methods. The computations have been done using the software packages FEniCS.

In our numerical example, the convergence order is computed by the following formula: order \(\simeq \frac{\log(E_i/E_{i+1})}{\log(h_i/h_{i+1})}\), where \(i\) responds to the spatial partition and \(E_i\) denotes the state and costate approximation and \(L^2(\Omega)\)-norm for the control approximation.

Example 6.1. For this example, the data under testing are as follows:

\[
\begin{aligned}
\text{minimize } & \left\{ \frac{1}{2} \left( \|\tilde{y}\|^2 + \|p\|^2 + \|y - y_d\|^2 + v\|u\|^2 \right) \right\}, \\
\text{div } p &= \tilde{y}, \quad x \in \Omega, \\
\text{div } p &= f + u, \quad x \in \Omega, \\
y &= \tilde{y} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

We now briefly describe the computational processes to be used for solving the numerical example for the following algorithm given below:

**Step 1:** Initialization: choose \(y_0, u_0\) and \(\mu_0 \in \mathbb{R}^N\) and set \(n=1\).

**Step 2:** Determine the active and inactive sets:

\[
A_n^a = \{ x : u_{n-1}(x) + \mu_{n-1}(x) < u_a(x) \}, \\
A_n^b = \{ x : u_{n-1}(x) + \mu_{n-1}(x) > u_b(x) \}, \\
I_n = \Omega \setminus (A_n^a \cap A_n^b).
\]

**Step 3:** If \(A_n^a = A_{n-1}^a\) and \(A_n^b = A_{n-1}^b\), then STOP, else.
Step 4: Determine the solution \((p, \tilde{p}, y, \tilde{y}, u) \in (V \times W)^2 \times U_{ad}\) such that
\[
(\tilde{p}, \nu) - (y, \text{div} \nu) = 0, \quad \forall \nu \in V, \\
(\text{div} \tilde{p}, w) = (\tilde{y}, w), \quad \forall w \in W, \\
(p, \nu) - (\tilde{y}, \text{div} \nu) = 0, \quad \forall \nu \in V, \\
(\text{div} p, w) = (f + u, w), \quad \forall w \in W, \\
(q, \nu) - (z, \text{div} \nu) = 0, \quad \forall \nu \in V, \\
(\text{div} q, w) = (y - y_d, w), \quad \forall w \in W.
\]

Further we define
\[
u(x) = \begin{cases} u_a(x) & \text{if } u(x) + \mu(x) < a(x), \\
-v^{-1}z(x) & \text{if } u(x) + \mu(x) \in [a(x), b(x)], \\
u_b(x) & \text{if } u(x) + \mu(x) > b(x). \end{cases}
\]

Step 5: Set \(\mu_n = -(v^{-1}z_n + u_n)\), update \(n = n + 1\) and continue with Step 2.

The existence of \((p, \tilde{p}, y, \tilde{y}, u)\) satisfying of step 4 the algorithm follows from the fact that it constitutes the optimality system for the auxiliary Problem 6.1 which has \((p_h, \tilde{p}_h, y_h, \tilde{y}_h, u_h)\) as unique solution.

We consider the following optimal state functions, costate functions for the fourth order elliptic partial differential equations:
\[
y = \sin(\pi x_1) \sin(\pi x_2), \quad z = -2\pi^2 v \sin(\pi x_1) \sin(\pi x_2), \\
u_f(x_1, x_2) = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2), \quad \tilde{y} = 2\pi^2 y.
\]

Due to the state equations (6.1), we obtain for the exact optimal control function \(u\):
\[
u(x_1, x_2) = \begin{cases} a, & \text{if } u_f(x_1, x_2) < a, \\
u_f(x_1, x_2), & \text{if } u_f(x_1, x_2) \in [a, b], \\
b, & \text{if } u_f(x_1, x_2) > b. \end{cases}
\]

\[f = \begin{cases} u_f(x_1, x_2) - a, & \text{if } u_f(x_1, x_2) < a, \\
0, & \text{if } u_f(x_1, x_2) \in [a, b], \\
u_f(x_1, x_2) - b, & \text{if } u_f(x_1, x_2) > b. \end{cases}
\]

\[\tilde{p} = -\left( \frac{\pi \cos(\pi x_1) \sin(\pi x_2)}{\pi \sin(\pi x_1) \cos(\pi x_2)} \right), \quad p = -\left( \frac{(\pi - \pi^3 v) \cos(\pi x_1) \sin(\pi x_2)}{(\pi - \pi^3 v) \sin(\pi x_1) \cos(\pi x_2)} \right).
\]

Thus the desired state variables are given by
\[y_d = (1 + 2\pi^4 v) \sin(\pi x_1) \sin(\pi x_2).\]
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Table 1: Numerical results for control and state variables on uniform meshes.

| $h$    | $||u - u_h||$ | $||y - y_h||$ | $||\tilde{y} - \tilde{y}_h||$ | $||p - p_h||$ | $||\tilde{p} - \tilde{p}_h||$ |
|--------|---------------|---------------|-------------------------------|---------------|-------------------------------|
| $16 \times 16$ | 3.123e-03   | 6.237e-04   | 1.642e-02   | 4.130e-02   | 2.465e-03   |
| $32 \times 32$ | 8.452e-04   | 1.486e-04   | 3.804e-03   | 1.139e-02   | 6.247e-04   |
| $64 \times 64$ | 2.599e-04   | 3.542e-05   | 8.861e-04   | 3.425e-03   | 1.847e-04   |
| $128 \times 128$ | 6.297e-05   | 8.890e-06   | 2.627e-04   | 8.812e-04   | 4.875e-05   |

Figure 2: Convergence plot for uniformly refined meshes.

In numerical implementation, we set $a = 6$ and $b = 16$ to make both the lower and the upper constraints active. Numerical solutions of uniformly refined meshes are presented in Table 1 respectively. We also show the convergence order by slopes in Fig 1.

Example 6.2. In this example, the test problem is as follows:

$$\min_{u \in U_{ad} \subset L^2(\Omega)} \left\{ \frac{1}{2} \left( ||\tilde{y}||^2 + ||\tilde{p}||^2 + ||y - y_d||^2 + v||u||^2 \right) \right\},$$

subject to the state equation

$$\begin{cases}
\text{div} \tilde{p} = \tilde{y}, & \tilde{p} = -\nabla y, & x \in \Omega, \\
\text{div} p = f + u, & p = -\nabla \tilde{y}, & x \in \Omega, \\
y = \tilde{y} = 0, & x \in \partial\Omega.
\end{cases}$$
Table 2: Numerical results for exact and approximate solutions.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u - u_h|$</th>
<th>$|y - y_h|$</th>
<th>$|\tilde{y} - \tilde{y}_h|$</th>
<th>$|p - p_h|$</th>
<th>$|\tilde{p} - \tilde{p}_h|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \times 16$</td>
<td>3.094e-03</td>
<td>8.040e-03</td>
<td>2.371e-02</td>
<td>3.100e-02</td>
<td>3.105e-02</td>
</tr>
<tr>
<td>$32 \times 32$</td>
<td>9.264e-04</td>
<td>2.060e-03</td>
<td>6.102e-03</td>
<td>8.080e-03</td>
<td>8.121e-03</td>
</tr>
<tr>
<td>$64 \times 64$</td>
<td>2.510e-04</td>
<td>5.264e-04</td>
<td>1.571e-03</td>
<td>2.041e-03</td>
<td>2.135e-03</td>
</tr>
</tbody>
</table>

Figure 3: Convergence plot for control and state variables.

Here we choose the domain $\Omega = [0, 1] \times [0, 1]$, $v = 1$ and $k = 1$. We consider the following optimal state functions, costate functions for the fourth order elliptic partial differential equations:

$$y = \sin(\pi x_1) \sin(\pi x_2), \quad z = \sin(2\pi x_1) \sin(2\pi x_2), \quad \tilde{y} = 2\pi^2 y.$$  

Due to the state equations (6.3)-(6.3), we obtain for the exact optimal control function $u$:

$$U_{ad} = \{ u \in L^2(\Omega) : 750 \leq u(x) \leq 50 \text{ a.e in } \Omega \}.$$  

Then the projection operator becomes

$$u = \max(a(x), \min(b(x), p)), \text{ for every } x \in \Omega.$$
Further we define
\[ \tilde{p} = - \begin{pmatrix} \pi \cos(\pi x_1) \sin(\pi x_2) \\ \pi \sin(\pi x_1) \cos(\pi x_2) \end{pmatrix}, \quad p = - \begin{pmatrix} 2\pi^3 \cos(\pi x_1) \sin(\pi x_2) \\ 2\pi^3 \sin(\pi x_1) \cos(\pi x_2) \end{pmatrix}, \]
\[ f = 4\pi^4 y - u, \quad y_d = y - 64\pi^4 z + 4\pi^4 y - 2\pi^2 y. \]

The domain \( \Omega \) is divided into regular uniform closed triangles. We will investigate the order of convergence of state, costate and control variables in \( L^2 \)-norm and the order of convergence of state and costate variables in the broken norm \( \| \cdot \| \). The errors of the numerical approximations to control and state variables on uniform meshes are listed in Table 2 respectively. We also show the convergence order by slopes in Fig 2.

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