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A PARAMETER-UNIFORM TAILORED FINITE POINT METHOD FOR SINGULARLY PERTURBED LINEAR ODE SYSTEMS*

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Abstract

In scientific applications from plasma to chemical kinetics, a wide range of temporal scales can present in a system of differential equations. A major difficulty is encountered due to the stiffness of the system and it is required to develop fast numerical schemes that are able to access previously unattainable parameter regimes. In this work, we consider an initial-final value problem for a multi-scale singularly perturbed system of linear ordinary differential equations with discontinuous coefficients. We construct a tailored finite point method, which yields approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters, to the exact solution. A parameter-uniform error estimate in the maximum norm is also proved. The results of numerical experiments, that support the theoretical results, are reported.

Mathematics subject classification: 37M05, 65G99.
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1. Introduction

We consider the following initial-final value problem for a system of linear ordinary differential equations with discontinuous coefficients

$$\mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{f}(t), \quad \forall t \in (p_k, p_{k+1}), \quad k = 0, \dots, K, \tag{1.1}$$

$$\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0) = 0, \quad k = 1, \dots, K, \tag{1.2}$$

$$\mathcal{B}^\varepsilon \mathbf{u}(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\,\mathbf{u}(1) = \mathbf{d}, \tag{1.3}$$

where $\mathcal{E}$, $\mathcal{B}^\varepsilon$ are $n \times n$ matrices, $\mathbf{d}$ is a vector, $\mathcal{A}(t)$ is an $n \times n$ matrix function and $\mathbf{f}(t)$ is a vector function on the interval $[0, 1]$ such that

$$\mathcal{A}(t) = \big(a_{i,j}(t)\big)_{n\times n}, \tag{1.4}$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T, \tag{1.5}$$

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and $\{p_k\}_0^{K+1}$ are some numbers which satisfy

$$0 = p_0 < p_1 < \cdots < p_K < p_{K+1} = 1.$$

The given functions $a_{i,j}(t), f_i(t)$ $(1 \leq i, j \leq n)$ may not be continuous on the whole interval $[0, 1]$. Here, we consider the case when they are piecewise continuous. More precisely, we assume that the functions $a_{i,j}(t), f_i(t)$ $(1 \leq i, j \leq n)$ have K points of discontinuity of the first kind, $t = p_k, (0 < p_k < 1; k = 1, \dots, K)$, so that on each subinterval $(p_k, p_{k+1}),$ $(k = 0, \dots, K; p_0 = 0, p_{K+1} = 1)$ the functions are smooth and satisfy the conditions

$$a_{i,i}(t) - \sum_{j=1, j\neq i}^{n} |a_{i,j}(t)| \geq \beta > 0, \quad \forall t \in [0, 1]. \tag{1.6}$$

Furthermore, we assume that the matrices $\mathcal{E}^\varepsilon = diag(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\mathcal{B}^\varepsilon = diag(b_1, b_2, \dots, b_n)$ are diagonal and satisfy the conditions

$$|\varepsilon_i| > 0, \;\; 1 \leq i \leq n, \tag{1.7}$$

$$b_i = \begin{cases} 1, & \varepsilon_i > 0, \\ 0, & \varepsilon_i < 0. \end{cases} \tag{1.8}$$

We also suppose that there exists at least one ε_j $(1 \leq j \leq n)$ such that

$$0 < |\varepsilon_j| \ll 1. \tag{1.9}$$

Problem (1.1)-(1.3) is then an initial-final value problem for a multi-scale singularly perturbed system of linear ordinary differential equations with discontinuous coefficients. The solution of problem (1.1)-(1.3) may contain initial, final and interior layers at any of the points p_k $(k = 1, \dots, K)$. The main goal in this paper is to develop a class of numerical methods, which yield approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters, to the exact solution of this problem.

When all of the parameters $(\varepsilon_j, \; j = 1, \dots, n)$ are positive, problem (1.1)-(1.3) reduces to the initial value singularly perturbed problem

$$\mathcal{L}\mathbf{u}(t) \equiv \mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{f}(t), \quad \forall t \in (p_k, \, p_{k+1}), \quad k = 0, \dots, K, \tag{1.10}$$

$$\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0) = 0, \qquad k = 1, \dots, K, \tag{1.11}$$

$$\mathbf{u}(0) = \mathbf{d}, \tag{1.12}$$

which has been studied in [23].They proposed a Shishkin piecewise uniform mesh with a classical finite difference scheme to obtain numerical solutions of this problem; a parameter-uniform error estimate was also given.

To motivate the study of the more general initial-final value problem in the paper, it should be noted that a semi-discretization, with respect to variable x, of the following forward-backward parabolic problem

$$\text{sign}(x)|x|^p \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sigma(x)u(x) + q(x,t), \quad -1 < x < 2, \, 0 < t < T, \tag{1.13}$$

$$u\big|_{x=-1} = f(t), \qquad u\big|_{x=2} = g(t), \quad 0 < t < T, \tag{1.14}$$

$$u\big|_{t=0} = s(x), \;\; 0 < x < 2, \qquad u\big|_{t=1} = \gamma(x), \;\; -1 < x < 0, \tag{1.15}$$

with $p > 1$, $\sigma(x) > 0$ can lead to the initial-final singularly perturbed problem (1.1)-(1.3). This is because $|x|^p \in (0, 2^p)$ ranges from very small to 2^p.

In this paper we construct a parameter-uniform scheme for the initial-final multi-scale singularly perturbed problem(1.1)-(1.3) using the tailored finite point method (TFPM). The TFPM was proposed by Han, Huang and Kellogg in [9] for the numerical solution of singular perturbation problems. The basic idea of the TFPM is to choose, at each mesh point, suitable basis functions based on the local properties of the solutions of the given problem; then to approximate the solution using these basis functions. At each point, the numerical scheme is tailored to the given problem. The TFPM was successfully applied by Han, Huang ,and Kellogg to solve the Hemker problem [9, 11]; they won the Hemker prize at the international conference BAIL 2008. Later the TFPM was developed to solve the second-order elliptic singular pertubation problem [6, 21], the first order wave equation [13], the one-dimension Helmholtz equation with high wave number [5], second order elliptic equations with rough or highly oscillatory coefficients [10] and so on [7,8]. For the one dimensional singular perturbation problem the TFPM is close to the method of "exponential fitting" discussed in [1,3,16,20]. The TFPM is also applied to the one-dimensional discrete-ordinate transport equations in [17].

The TFPM uses the functions that exactly satisfy the PDE as the bases. For the linear ordinary differential equations system under consideration, the bases are exponential functions. The exponential integrator appears for a long time, see the review paper [19]. The new idea in TFPM is to approximate all the coefficients $\mathcal{A}(t)$ and $\mathbf{f}(t)$ by piecewise contants. Compared with previous magnus integrators [12] and adiabatic integrators [18] developed for the stiff problem or highly oscillatory problems, our approach is simple and proved to possess parameter uniform convergence. More precisely, we can use time steps much larger than the parameters ϵ_i in (1.7) and achieve stable and accurate results.

The main contribution of this paper is that we construct a tailored finite point method for (1.1)-(1.3), which yields approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters, to the exact solution. The parameter-uniform error estimate in the maximum norm is proved analytically as stated in Theorem 4.1. Some numerical experiments that support the theoretical results are presented in section 5.

2. Existence and Uniqueness of Solutions to(1.1)-(1.3)

In this section we discuss the existence and uniqueness of solutions to the problem (1.1)-(1.3). On $[0,1]$ we introduce the following function spaces

$$\mathcal{C}^{(0)}_{*,*}[0,1] = \Big\{v(t)\big|v|_{(p_k,p_{k+1})} \in \mathcal{C}^{(0)}(p_k,p_{k+1}), \quad k=0,1,\dots,K\Big\}, \tag{2.1}$$

$$\mathcal{C}^{(1)}_{*}[0,1], = \Big\{v(t)\big|v(t) \in \mathcal{C}^{(0)}[0,1], and\ v'(t) \in \mathcal{C}^{(0)}_{**}[0,1]\Big\}, \tag{2.2}$$

$$\mathcal{C}^{(1)}_{*,*}[0,1] = \Big\{v(t)\big|v(t), v'(t) \in \mathcal{C}^{(0)}_{**}[0,1]\Big\}. \tag{2.3}$$

Note that the space $\mathcal{C}^{(0)}[0,1]$ is a subspace of $\mathcal{C}^{(0)}_{**}[0,1]$ and that, for any $v(t) \in \mathcal{C}^{(0)}_{*,*}[0,1]$, $v(t)$ may have discontinuity points of the first kind $t = p_k$ $(k = 1,\dots,K)$. On each subinterval $[p_k,\ p_{k+1}]$ we adopt the conventions that $v(p_k) = \lim_{t\to p_k^+} v(p_k),\ v(p_{k+1}) = \lim_{t\in p_{k+1}^-} v(p_{k+1})$. The space $\mathcal{C}^{(1)}_{*}[0,1]$ is a subspace of $\mathcal{C}^{(1)}_{*,*}[0,1]$.

For any $v(t) \in \mathcal{C}^{(0)}_{*,*}[0,1]$, we define the norm

$$\|v\|_\infty = \max_{0\le k\le K}\Big\{\max_{p_k\le t\le p_{k+1}}|v(t)|\Big\},\tag{2.4}$$

and for any $v(t) \in \mathcal{C}^{(1)}_{*,*}[0,1]$, we define the norms

$$|v|_{1,\infty} = \max_{0\le k\le,K}\Big\{\max_{p_k\le t\le p_{k+1}}|v'(t)|\Big\},\tag{2.5}$$

$$\|v\|_{1,\infty} = \max\Big\{\|v\|_\infty, |v|_{1,\infty}\Big\}.\tag{2.6}$$

Furthermore, we introduce the following function vector and matrix spaces

$$\mathbb{C}^{(1)}_{*}[0,1] = \big(\mathcal{C}^{(1)}_{*}[0,1]\big)^n,\tag{2.7}$$

$$\mathbb{C}^{(1)}_{*,*}[0,1] = \big(\mathcal{C}^{(1)}_{*,*}[0,1]\big)^n,\tag{2.8}$$

$$\mathfrak{C}^{(1)}_{*,*}[0,1] = \big(\mathcal{C}^{(1)}_{*,*}[0,1]\big)^{n\times n}.\tag{2.9}$$

Note that $\mathbb{C}^{(1)}_{*}[0,1]$ is a subspace of $\mathbb{C}^{(1)}_{*,*}[0,1]$. For any $\mathbf{v}(t) \in \mathbb{C}^{(1)}_{*,*}[0,1]$, we define the norms $\|\cdot\|_\infty$, $|\cdot|_{1,\infty}$, $\|\cdot\|_{1,\infty}$

$$\|\mathbf{v}\|_\infty = \max_{0\le k\le K}\Big(\max_{p_k\le t\le P_{k+1}}\|\mathbf{v}(t)\|_\infty\Big),\tag{2.10}$$

$$|\mathbf{v}|_{1,\infty} = \|\mathbf{v}'\|_\infty,\tag{2.11}$$

$$\|\mathbf{v}\|_{1,\infty} = \max\big(\|\mathbf{v}\|_\infty, \|\mathbf{v}'\|_\infty\big),\tag{2.12}$$

where the norm $\|\mathbf{v}(t)\|_\infty$ is the norm in the vector space $\mathbf{R}^n$ for fixed $t \in [0,1]$. Similarly we define the norms $\|\cdot\|_\infty$, $|\cdot|_{1,\infty}$, $\|\cdot\|_{1,\infty}$ in the space $\mathfrak{C}^{(1)}_{*,*}[0,1]$.

For problem (1.1)-(1.3), we then have the following theorem.

Theorem 2.1. *Suppose that $\mathcal{A}(t) \in \mathfrak{C}^{(0)}_{*,*}[0,1]$, $\mathbf{f}(t) \in \mathbb{C}^{(0)}_{*,*}[0,1]$ and $\mathcal{A}(t), \mathbf{f}(t)$ satisfy the conditions* (1.4)-(1.9). *Then, for problem* (1.1)-(1.3) *there exists a unique solution $\mathbf{u}^\varepsilon(t)$, $\mathbf{u}^\varepsilon(t) \in \mathbb{C}^{(1)}_{*}[0,1]$, and the following estimate holds*

$$\|\mathbf{u}^\varepsilon\|_\infty \le \max\Big\{\|\mathbf{d}\|_\infty, \frac{\|\mathbf{f}\|_\infty}{\beta}\Big\}.\tag{2.13}$$

Proof. We first establish the estimate (2.13). Suppose that $\mathbf{u}^\varepsilon(t) = (u^\varepsilon_1(t), \ldots, u^\varepsilon_n(t))^T \in \mathbb{C}^{(1)}_{*}[0,1]$ is a solution of problem (1.1)-(1.3). Let

$$M = \|\mathbf{u}^\varepsilon\|_\infty.\tag{2.14}$$

By the continuity of $\mathbf{u}^\varepsilon(t)$ on the interval $[0,1]$, we can find a point $t_m \in [0,1]$, such that

$$M = |u^\varepsilon_i(t_m)|,$$

for some integer i , $1 \le i \le n$. If $M = 0$, the estimate follows directly. Otherwise, we consider the case $u^\varepsilon_i(t_m) > 0$ (for the case $u^\varepsilon_i(t_m) < 0$, the proof is similar). Then we know that

$$|u^\varepsilon_j(t)| \le u^\varepsilon_i(t_m) = M, \quad \forall t \in [0,1], \quad 1 \le j \le n.\tag{2.15}$$

(i) If $t_m \ne p_k$, $\quad k = 0, 1, \ldots, K+1$, the inequality (2.15) yields

$$u^{\varepsilon\,\prime}_i(t_m) = 0.\tag{2.16}$$

From system (1.1), we obtain

$$a_{i,i}(t_m)u_i^\varepsilon(t_m) + \sum_{1\le j\le n, j\ne i} a_{i,j}(t_m)u_j^\varepsilon(t_m) = f_i(t_m), \tag{2.17}$$

and the estimate follows immediately from (1.6).
(ii) If $t_m = p_k, \quad k = 0, 1, \ldots, K+1$, suppose that $\varepsilon_i > 0$ (for the case $\varepsilon_i < 0$, the proof is similar), If $k = 0$, then $t_m = 0$, and so, from the initial- final condition (1.3), we know that $M = u_i^\varepsilon(t_m) = d_i$, which yields the estimate (2.13). Otherwise, if $0 < k \le K+1$, note that, on the interval $[p_{k-1}, p_k]$, the function $u_i^\varepsilon(t)$ satisfies system (1.1) at the the point $t = p_k$, and so

$$a_{i,i}(t_m - 0)u_i^\varepsilon(t_m) + \sum_{1\le j\le n, j\ne i} a_{i,j}(t_m - 0)u_j^\varepsilon(t_m) \le f_j(t_m - 0). \tag{2.18}$$

This completes the proof of the estimate (2.13).

Using estimate (2.13), we can now show that Problem (1.1)-(1.3) has a unique solution. To see this we note that the following homogeneous initial-final value problem

$$\mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{0}, \qquad \forall t \in (p_k,\, p_{k+1}), \quad k = 0, \ldots, K, \tag{2.19}$$

$$\big[\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0)\big] = \mathbf{0}, \quad k = 1, \ldots, K, \tag{2.20}$$

$$\mathcal{B}^\varepsilon\mathbf{u}(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\,\mathbf{u}(1) = \mathbf{0}, \tag{2.21}$$

has a unique solution $\mathbf{u}^\varepsilon(t) = \mathbf{0}$.

To prove the existence of a solution to problem (1.1)-(1.3), we introduce the following auxiliary initial value problems

$$\mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{f}(t), \qquad \forall t \in (p_k,\, p_{k+1}), \quad k = 0, \ldots, K, \tag{2.22}$$

$$\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0) = \mathbf{0}, \quad k = 1, \ldots, K, \tag{2.23}$$

$$\mathbf{u}(0) = \mathbf{0} \tag{2.24}$$

and

$$\mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{0}, \qquad \forall t \in (p_k,\, p_{k+1}), \quad k = \mathbf{0}, \ldots, K, \tag{2.25}$$

$$\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0) = \mathbf{0}, \quad k = 1, \ldots, K, \tag{2.26}$$

$$\mathbf{u}(0) = \mathbf{e}_j, \tag{2.27}$$

for each integer j $(1 \le j \le n)$.

By the existence theorem for the initial value problem for the linear ODE system, we know that for problem (2.22)-(2.24) there exists a unique solution $\mathbf{v}_f^\varepsilon(t) = (v_{1,f}^\varepsilon(t), \ldots, v_{n,f}^\varepsilon(t))^T \in \mathbb{C}_*^{(1)}[0,1]$, and for each integer $1 \le j \le n$, for problem (2.25)-(2.27) there exists a unique solution $\mathbf{v}_j^\varepsilon(t) = (v_{j,1}^\varepsilon(t), \ldots, v_{j,n}^\varepsilon(t))^T \in \mathbb{C}_*^{(1)}[0,1]$. The general solutions of system (1.1) are given by

$$\mathbf{u}^\varepsilon(t) = \sum_{j=1,\ldots,n} c_j\mathbf{v}_j^\varepsilon(t) + \mathbf{v}_f^\varepsilon(t), \tag{2.28}$$

for arbitrary constants $c_j, (j = 1, \ldots, n)$. Let $\mathbf{c} = (c_1, \ldots, c_n)^T$ and

$$\mathbf{V}^\varepsilon(t) = (\mathbf{v}_1^\varepsilon(t), \ldots, \mathbf{v}_n^\varepsilon(t)), \tag{2.29}$$

where $\mathbf{V}^\varepsilon(t)$ is an $n \times n$ matrix function. Then equality (2.28) can be rewritten as follows

$$\mathbf{u}^\varepsilon(t) = \mathbf{V}^\varepsilon(t)\mathbf{c} + \mathbf{v}_f^\varepsilon(t). \tag{2.30}$$

The vector-valued function $\mathbf{u}^\varepsilon(t)$ given by (2.30) satisfies system (1.1) and the continuity condition (1.2) for any constant vector $\mathbf{c}$. Thus, if we can find a vector $\mathbf{c} \in \mathbf{R}^n$, such that $\mathbf{u}^\varepsilon(t)$ satisfies the initial-final condition (1.3), the existence of a solution to problem (1.1)-(1.3) has been proved. The initial-final condition (1.3) yields

$$\big[\mathcal{B}^\varepsilon \mathbf{V}^\varepsilon(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\,\mathbf{V}^\varepsilon(1)\big]\mathbf{c} = \mathbf{d} - \big[\mathcal{B}^\varepsilon \mathbf{v}_f^\varepsilon(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\,\mathbf{V}_f^\varepsilon(1)\big]. \tag{2.31}$$

The uniqueness of the solution of problem (1.1)-(1.3) leads to

$$\det\Big(\mathcal{B}^\varepsilon \mathbf{V}^\varepsilon(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\,\mathbf{V}^\varepsilon(1)\Big) \neq \mathbf{0}. \tag{2.32}$$

This implies that for system (2.31) there exists a unique solution $\mathbf{c} \in \mathbf{R}^n$. Then, from (2.30), we obtain a solution of problem (1.1)-(1.3). □

3. An Approximation to Problem (1.1)-(1.3)

In this section we construct an approximation to problem (1.1)-(1.3). The interval $[0,1]$ is divided into subintervals by

$$0 = t_0 < t_1 < \cdots < t_L = 1, \tag{3.1}$$

such that, for each $p_k \in [0,1]$, we can find a t_l such that $p_k = t_l$. Furthermore, let

$$h_l = t_l - t_{l-1}, \quad \forall\, l = 1, \ldots, L; \quad h = \max_{l=1,\ldots,L}(h_l). \tag{3.2}$$

On each subinterval (t_{l-1}, t_l) the functions $a_{i,j}(t), f_i(t)$ are approximated by the constants $a_{i,j}(t_l^*)$, and $f_i(t_l^*)$ with $t_{l-1} \le t_l^* \le t_l$. Then we introduce the matrix and vector

$$\mathcal{A}^l = \Big(a_{i,j}(t_l^*)\Big)_{n\times n}, \tag{3.3}$$

$$\mathbf{f}^l = (f_1(t_l^*), f_2(t_l^*), \ldots, f_n(t_l^*))^T. \tag{3.4}$$

We also define the approximating matrix and vector functions

$$\mathcal{A}_h(t) = \mathcal{A}^l, \quad \forall t \in (t_{l-1}, t_l), \tag{3.5}$$

$$\mathbf{f}_h(t) = \mathbf{f}^l, \quad \forall t \in (t_{l-1}, t_l). \tag{3.6}$$

It is easy to see that for these approximating functions, the following estimates hold

$$\|\mathbf{f}_h\|_\infty \le \|\mathbf{f}\|_\infty, \tag{3.7}$$

$$\|\mathbf{f} - \mathbf{f_h}\|_\infty \equiv \max_{l=0,1,\ldots,L}\ \max_{t_{l-1}\le t\le t_l} \|\mathbf{f}(t) - \mathbf{f_h}(t)\|_\infty \le \|\mathbf{f}'\|_\infty h, \tag{3.8}$$

$$\|\mathcal{A} - \mathcal{A}_h\|_\infty \equiv \max_{l=0,1,\ldots,L}\ \max_{t_{l-1}\le t\le t_l} \|\mathcal{A}(t) - \mathcal{A}_h(t)\|_\infty \le c\|\mathcal{A}'\|_\infty h, \tag{3.9}$$

where c is a constant, independent of h.

Now consider the approximate problem

$$\mathcal{E}\mathbf{u}_h^{\varepsilon\prime}(t) + \mathcal{A}_h(t)\mathbf{u}_h^\varepsilon(t) = \mathbf{f}_h(t), \quad \forall t \in (t_{l-1},\, t_l), \quad l = 1, \ldots, K, \tag{3.10}$$

$$\mathbf{u}_h^\varepsilon(t_l + 0) - \mathbf{u}_h^\varepsilon(t_l - 0) = 0, \qquad l = 1, \ldots, L-1, \tag{3.11}$$

$$\mathcal{B}^\varepsilon \mathbf{u}_h^\varepsilon(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\, \mathbf{u}_h^\varepsilon(1) = \mathbf{d}. \tag{3.12}$$

By Theorem 2.1, we know that for problem (3.10)-(3.12) there exists a unique solution $\mathbf{u}_h^\varepsilon(t)$, and that the following estimate holds

$$\|\mathbf{u}_h^\varepsilon\|_\infty \leq \max\Big(\|\mathbf{d}\|_\infty, \frac{\|\mathbf{f}\|_\infty}{\beta}\Big). \tag{3.13}$$

We introduce the error function $\mathbf{r}_h^\varepsilon(t) = \mathbf{u}^\varepsilon(t) - \mathbf{u}_h^\varepsilon(t)$. Then $\mathbf{r}_h^\varepsilon(t)$ satisfies

$$\mathcal{E}\mathbf{r}_h^{\varepsilon\prime}(t) + \mathcal{A}_h(t)\mathbf{r}_h^\varepsilon(t) = \mathbf{d}_h^\varepsilon(t), \quad \forall t \in (t_{l-1},\, t_l), \quad l = 1, \ldots, K, \tag{3.14}$$

$$\mathbf{r}_h^\varepsilon(t_l + 0) - \mathbf{r}_h^\varepsilon(t_l - 0) = 0, \qquad l = 1, \ldots, L-1, \tag{3.15}$$

$$\mathcal{B}^\varepsilon \mathbf{r}_h^\varepsilon(0) + (\mathbf{I} - \mathcal{B}^\varepsilon)\, \mathbf{r}_h^\varepsilon(1) = \mathbf{0}, \tag{3.16}$$

with

$$\mathbf{d}_h^\varepsilon(t) = \mathbf{f}(t) - \mathbf{f}_h(t) + \big(\mathcal{A}_h(t) - \mathcal{A}(t)\big)\mathbf{u}_h^\varepsilon(t). \tag{3.17}$$

Combining equality (3.17) and inequalities (3.7)-(3.9),(3.13), we arrive at

$$\|\mathbf{d}_h^\varepsilon\|_\infty \leq \bigg(\|\mathbf{f}'\|_\infty + c\|\mathcal{A}'\|_\infty\Big(\|\mathbf{d}\|_\infty + \frac{\|\mathbf{f}\|_\infty}{\beta}\Big)\bigg)h, \tag{3.18}$$

where c is a constant independent of h and ε.

Applying Theorem 2.1 to problem (3.14)-(3.16) yields the following error bound.

Theorem 3.1. *The error $\mathbf{r}_h^\varepsilon$ satisfies the estimate*

$$\|\mathbf{r}_h^\varepsilon\|_\infty \leq \frac{1}{\beta}\Big\{\|\mathbf{f}'\|_\infty + c\|\mathcal{A}'\|_\infty\big(\|\mathbf{d}\|_\infty + \frac{\|\mathbf{f}\|_\infty}{\beta}\big)\Big\}h, \tag{3.19}$$

where c is a constant independent of h and $\mathcal{E}$.

This shows that the approximate solution $\mathbf{u}_h^\varepsilon(t)$ converges $\mathcal{E}$-uniformly to the solution $\mathbf{u}^\varepsilon(t)$ of problem (1.1)-(1.3) on the interval $[0, 1]$.

4. A Tailored Finite Point Method for Problem (1.1)–(1.3)

Using $\mathbf{u}_h^\varepsilon(t)$ for the solution of Problem (3.9)–(3.11), we now introduce the tailored finite point method for problem (1.1)–(1.3). Let

$$\mathbf{u}_h^\varepsilon(t_l) = \mathbf{u}_l, \qquad l = 0, 1, \ldots, L. \tag{4.1}$$

Then on each subinterval $[t_{l-1}, t_l]$,the approximate solution $\mathbf{u}_h^\varepsilon(t)$ satisfies

$$\mathcal{E}\mathbf{u}_h^{\varepsilon\prime}(t) + \mathcal{A}^l\mathbf{u}_h^\varepsilon(t) = \mathbf{f}^l, \quad \forall t \in (t_{l-1},\, t_l), \tag{4.2}$$

$$\mathcal{B}^\varepsilon \mathbf{u}_h^\varepsilon(t_{l-1}) + (\mathbf{I} - \mathcal{B}^\varepsilon)\, \mathbf{u}_h^\varepsilon(t_l) = \mathcal{B}^\varepsilon \mathbf{u}_{l-1} + (\mathbf{I} - \mathcal{B}^\varepsilon)\, \mathbf{u}_l. \tag{4.3}$$

It is easy to see that the ODE system (4.2) with constant coefficients has the particular solution

$$\mathbf{v}_f^l = \big(\mathcal{A}^l\big)^{-1}\mathbf{f}^l. \tag{4.4}$$

We construct the general solution $\mathbf{v}_h^\epsilon$ of the homogeneous ODE system corresponding to (4.2)

$$\mathcal{E}\mathbf{v}_h^{\varepsilon\,\prime}(t) + \mathcal{A}^l\mathbf{v}_h^\varepsilon(t) = \mathbf{0}, \quad \forall t \in (t_{l-1},\, t_l). \tag{4.5}$$

Let

$$\mathbf{v}_h^\varepsilon(t) = \xi e^{\lambda t} \tag{4.6}$$

be the solution of system (4.5). Then the vector ξ and number λ are the solutions of the eigenvalue problem

$$\mathcal{A}^l\xi = -\lambda\mathcal{E}\xi. \tag{4.7}$$

Solving the eigenvalue problem (4.7), we obtain the eigenvalues $\lambda_1^l, \lambda_2^l, \ldots, \lambda_{2r}^l, \lambda_{2r+1}^l, \lambda_n^l$, corresponding to the eigenvectors $\xi_1^l, \xi_2^l, \ldots, \xi_{2r}^l, \xi_{2r+1}^l, \ldots, \xi_n^l$. Assume that the eigenvalues λ_{2r+1}^l, $\ldots, \lambda_n^l$ and the corresponding eigenvectors are real. The remaining eigenvalues occur in complex conjugate pairs, so that

$$\lambda_{2j}^l = \overline{\lambda_{2j-1}^l}, \quad j = 1, \ldots, r, \tag{4.8}$$

$$\xi_{2j}^l = \overline{\xi_{2j-1}^l}, \quad j = 1, \ldots, r. \tag{4.9}$$

These conjugate pairs of eigenvalues $\lambda_{2j-1}^l, \lambda_{2j}^l$ and eigenvectors ξ_{2j-1}^l, ξ_{2j}^l, can be written in the form

$$\begin{cases} \lambda_{2j-1}^l = \alpha_j^l - i\beta_j^l, & \lambda_{2j}^l = \alpha_j^l + i\beta_j^l, \\ \xi_{2j-1}^l = \eta_j^l - i\zeta_j^l, & \xi_{2j}^l = \eta_j^l + i\zeta_j^l. \end{cases}$$

Then, for each $1 \le j \le r$, there are two independent solutions of the system (4.5) of the form

$$\mathbf{v}_{2j-1}^l(t) = \Big(\eta_j^l\cos(\beta_j^l t) - \zeta_j^l\sin(\beta_j^l t)\Big)e^{\alpha_j^l(t-t_l)}, \tag{4.10}$$

$$\mathbf{v}_{2j}^l(t) = \Big(\eta_j^l\cos(\beta_j^l t) + \zeta_j^l\sin(\beta_j^l t)\Big)e^{\alpha_j^l(t-t_l)}. \tag{4.11}$$

Furthermore, for each j $(2r < j \le n)$, there is one solution of the system (4.5) given by

$$\mathbf{v}_j^l(t) = \xi_j^l e^{\lambda_j^l(t-t_l)}. \tag{4.12}$$

Let

$$\mathcal{V}^l(t) \equiv \Big(\mathbf{v}_1^l(t), \mathbf{v}_2^l(t), \ldots, \mathbf{v}_n^l(t)\Big), \tag{4.13}$$

where $\mathcal{V}^l(t)$ is a $n \times n$ matrix function. On each interval $[t_{l-1},\, t_l]$, $\mathbf{u}_h^\varepsilon(t)$ is given by

$$\mathbf{u}_h^\varepsilon(t) = \mathcal{V}^l(t)\mathbf{c}^l + \mathbf{v}_f^l, \tag{4.14}$$

with $\mathbf{c}^l = (c_1^l, \ldots, c_n^l)^T$, which is determined by the initial-final condition (4.3). Then we have

$$\begin{aligned} \mathbf{c}^l &= \Big(\mathcal{B}^\varepsilon\mathcal{V}^l(t_{l-1}) + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathcal{V}^l(t_l)\Big)^{-1}\Big(\mathcal{B}^\varepsilon\mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big) \\ &\equiv \mathcal{D}^l\Big(\mathcal{B}^\varepsilon\mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big), \end{aligned} \tag{4.15}$$

with

$$\mathcal{D}^l = \Big(\mathcal{B}^\varepsilon \mathcal{V}^l(t_{l-1}) + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathcal{V}^l(t_l)\Big)^{-1}.$$

Furthermore, on each interval $[t_{l-1},\, t_l]$, the expression (4.14) can be written as

$$\mathbf{u}_h^\varepsilon(t) = \mathcal{V}^l(t)\mathcal{D}^l\Big(\mathcal{B}^\varepsilon \mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big) + \mathbf{v}_f^l. \tag{4.16}$$

The continuity condition (3.12) yields

$$\big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_{l-1} = \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\Big\{\mathcal{V}^l(t_{l-1})\mathcal{D}^l\Big(\mathcal{B}^\varepsilon \mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big) + \mathbf{v}_f^l\Big\}, \tag{4.17}$$

$$\mathcal{B}^\varepsilon \mathbf{u}_l = \mathcal{B}^\varepsilon\Big\{\mathcal{V}^l(t_l)\mathcal{D}^l\Big(\mathcal{B}^\varepsilon \mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big) + \mathbf{v}_f^l\Big\}. \tag{4.18}$$

We then obtain the following tailored finite point scheme

$$\begin{aligned}\mathbf{u}_l = {}& \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\Big\{\mathcal{V}^{l+1}(t_l)\mathcal{D}^{l+1}\Big(\mathcal{B}^\varepsilon \mathbf{u}_l + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_{l+1} - \mathbf{v}_f^{l+1}\Big) + \mathbf{v}_f^{l+1}\Big\} \\ & + \mathcal{B}^\varepsilon\Big\{\mathcal{V}^l(t_l)\mathcal{D}^l\Big(\mathcal{B}^\varepsilon \mathbf{u}_{l-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_l - \mathbf{v}_f^l\Big) + \mathbf{v}_f^l\Big\},\ \ l = 1, 2, \dots, L-1,\end{aligned} \tag{4.19}$$

$$\mathbf{u}_0 = \mathcal{B}^\varepsilon \mathbf{d} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\Big\{\mathcal{V}^1(t_0)\mathcal{D}^1\Big(\mathcal{B}^\varepsilon \mathbf{d} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{u}_1 - \mathbf{v}_f^1\Big) + \mathbf{v}_f^1\Big\}, \tag{4.20}$$

$$\mathbf{u}_L = \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{d} + \mathcal{B}^\varepsilon\Big\{\mathcal{V}^L(t_L)\mathcal{D}^L\Big(\mathcal{B}^\varepsilon \mathbf{u}_{L-1} + \big(\mathbf{I} - \mathcal{B}^\varepsilon\big)\mathbf{d} - \mathbf{v}_f^L\Big) + \mathbf{v}_f^L\Big\}. \tag{4.21}$$

In the special case when the $\{\varepsilon_j,\ j = 1.\dots, n\}$ are all positive, we have $\mathcal{B}^\varepsilon = \mathbf{I}$ and the method (4.19)-(4.21), for finding the numerical solution of problem (1.10)-(1.12), is reduced to the following method

$$\mathbf{u}_l = \mathcal{V}^l(t_l)\mathcal{D}^l\big(\mathbf{u}_{l-1} - \mathbf{v}_f^l\big) + \mathbf{v}_f^l,\ \ l = 1, 2, \dots, L, \tag{4.22}$$

$$\mathbf{u}_0 = \mathbf{d}. \tag{4.23}$$

This is a one step explicit scheme, which is unconditionally stable.

Since the discrete scheme (4.19)-(4.21) is equivalent to the approximate problem (3.9)-(3.11), we attain the following result.

Theorem 4.1. *Suppose that* $\mathcal{A}(t) \in \mathfrak{C}^{(1)}_{*,*}[0,1]$, $\mathbf{f}(t) \in \mathbb{C}^{(1)}_{*,*}[0,1]$ *and that* $\mathcal{A}(t), \mathbf{f}(t)$ *satisfy the conditions* (1.3)-(1.9). *Then there exists a unique solution* $\{\mathbf{u}_l \quad l = 0, 1, \dots, L\}$ *to problem* (4.19)-(4.21) *and the following parameter-uniform error estimate holds*

$$\max_{l=0,1,\dots,L} \|\mathbf{u}(t_l) - \mathbf{u}_l\|_\infty \le \frac{1}{\beta}\Big(\|\mathbf{f}'\|_\infty + c\|\mathcal{A}'\|_\infty\Big(\|\mathbf{d}\|_\infty + \frac{\|\mathbf{f}\|_\infty}{\beta}\Big)\Big)h. \tag{4.24}$$

This shows that the tailored finite point method yields approximate solutions that are parameter-uniformly convergent with respect to $\{0 < |\varepsilon_j| \le 1, \quad \forall j = 1.\dots, n\}$ to the solution of the multi-scale singularly perturbed problem (1.1)-(1.3).

Solving the algebraic equation (4.19)-(4.21) gives the numerical solution $\{\mathbf{u}_l, \quad l = 0, 1, \dots, L\}$. Combining this with the expressions (4.15), (4.14) then yields the required continuous approximate solution $\mathbf{u}_h^\varepsilon(t)$.

5. Numerical Examples

In this section three problems are solved numerically using a tailored finite point method. The first is an initial value problem and the second an initial-final value problem for a singularly perturbed system of ordinary differential equations. The third problem arises from a semi-discretization of a forward-backward parabolic problem. The errors in the numerical solutions for the first two problems, as well as numerical estimates of the parameter-uniform rates of convergence and the parameter-uniform error constants, are given in five tables. In Tables 5.1, 5.3, 5.5 the errors of the numerical solutions are first calculated by comparing the numerical solutions with the "exact" solutions that are obtained with the finest mesh $\Delta t = 1/4096$. On the other hand, in Tables 5.2, 5.4 parameter-uniform estimates of the convergence rates and error constants are computed using the methodology described in [4]. Graphs of the solutions of all three problems are presented in the figures.

Example 1. For $t \in (0,1)$, we solve

$$\begin{aligned}
&\varepsilon_1 u_1'(t) + 4u_1(t) - u_2(t) - u_3(t) = t,\\
&\varepsilon_2 u_2'(t) - u_1(t) + 4u_2(t) - u_3(t) = 1,\\
&\varepsilon_3 u_3' - u_1(t) - u_2(t) + 4u_3(t) = 1 + t^2,\\
&u_1(0) = 0, \quad u_2(0) = 0, \quad u_3(0) = 0.
\end{aligned}$$

for different values of ε_1, ε_2 and ε_3. This example appears in [23] and we compare our tailored finite point method with the classical finite difference discretization using a Shishkin piecewise uniform mesh. Both methods are seen to be first order parameter-uniform. Similarly to [23], let $\varepsilon_1 = r/16$, $\varepsilon_2 = r/4$, $\varepsilon_3 = r$. The numerical solutions of $u_1(t)$ for different r are displayed in Fig. 5.1.

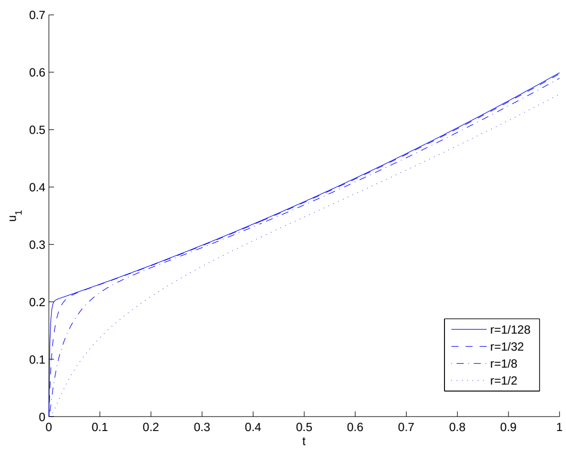


Fig. 5.1. Example 1. $u_1(t)$ for $r = 1/2, 1/8, 1/32, 1/128$.

The L^∞ norm of the numerical errors for different r and time step are given in Table 5.1, where we have used $t_l^* = t_{l-1}$ in the formulas (3.3) and (3.4). Parameter-uniform first order convergence for all r can be observed in this table,

Table 5.1: Example 1. L^∞ norm of the numerical errors, for different r and time steps, when $t_l^* = t_{l-1}$ in (3.3) and (3.4), $\varepsilon_1 = r/16$, $\varepsilon_2 = r/4$, $\varepsilon_3 = r$

r \ Δt	1/128	1/256	1/512	1/1024	1/2048
2^{-1}	$2.311 * 10^{-3}$	$1.115 * 10^{-3}$	$5.194 * 10^{-4}$	$2.224 * 10^{-4}$	$7.412 * 10^{-5}$
2^{-2}	$2.507 * 10^{-3}$	$1.205 * 10^{-3}$	$5.605 * 10^{-4}$	$2.398 * 10^{-4}$	$7.987 * 10^{-5}$
2^{-3}	$2.638 * 10^{-3}$	$1.258 * 10^{-3}$	$5.829 * 10^{-4}$	$2.489 * 10^{-4}$	$8.283 * 10^{-5}$
2^{-4}	$2.768 * 10^{-3}$	$1.300 * 10^{-3}$	$5.979 * 10^{-4}$	$2.544 * 10^{-4}$	$8.449 * 10^{-5}$
2^{-5}	$2.967 * 10^{-3}$	$1.354 * 10^{-3}$	$6.132 * 10^{-4}$	$2.589 * 10^{-4}$	$8.568 * 10^{-5}$
2^{-6}	$3.315 * 10^{-3}$	$1.448 * 10^{-3}$	$6.373 * 10^{-4}$	$2.649 * 10^{-4}$	$8.700 * 10^{-5}$
2^{-7}	$3.877 * 10^{-3}$	$1.619 * 10^{-3}$	$6.820 * 10^{-4}$	$2.757 * 10^{-4}$	$8.917 * 10^{-5}$
2^{-10}	$5.139 * 10^{-3}$	$2.418 * 10^{-3}$	$1.057 * 10^{-3}$	$4.010 * 10^{-4}$	$1.174 * 10^{-4}$
2^{-15}	$5.280 * 10^{-3}$	$2.559 * 10^{-3}$	$1.195 * 10^{-3}$	$5.124 * 10^{-4}$	$1.708 * 10^{-4}$
2^{-16}	$5.280 * 10^{-3}$	$2.559 * 10^{-3}$	$1.195 * 10^{-3}$	$5.124 * 10^{-4}$	$1.708 * 10^{-4}$
2^{-17}	$5.280 * 10^{-3}$	$2.559 * 10^{-3}$	$1.195 * 10^{-3}$	$5.124 * 10^{-4}$	$1.708 * 10^{-4}$

The parameter-uniform error parameters p^* and $C^*_{p^*}$ are given by the well-established two-mesh procedure for numerically finding a parameter-uniform error bound of the form

$$\|U_\epsilon^{\Delta t} - u_\epsilon\|_\infty \le C^*_{p^*} \Delta t^{p^*},$$

which is described in [4]. The technique uses the two-mesh method, which involves the quantities

$$D_\epsilon^{\Delta t} = \|U_\epsilon^{\Delta t} - U_\epsilon^{\Delta t/2}\|_\infty, \qquad D^{\Delta t} = \max_\epsilon \{D_\epsilon^{\Delta t}\},$$
$$p^{\Delta t} = \log_2 \frac{D^{\Delta t}}{D^{\Delta t/2}}, \qquad p^* = \min_{\Delta t} p^{\Delta t},$$
$$C_{p^*}^{\Delta t} = \frac{D^{\Delta t}}{\Delta t^{p^*}(1-2^{-p^*})}, \qquad C^*_{p^*} = \max_{\Delta t} C_{p^*}^{\Delta t}.$$

Table 5.2: Example 1. Values of $D_\epsilon^{\Delta t}$, $D^{\Delta t}$, $p^{\Delta t}$ and $C_{p^*}^{\Delta t}$ with $\varepsilon_1 = r/16$, $\varepsilon_2 = r/4$, $\varepsilon_3 = r$ for various r and Δt. We find $p^* = \min_{\Delta t} p^{\Delta t} = 0.996$ and $C^*_{p^*} = \max_{\Delta t} C^{\Delta t}_{p^*} = 0.685$. Here $t_l^* = t_{l-1}$ in (3.3) and (3.4).

r \ Δt	1/128	1/256	1/512	1/1024	1/2048
2^{-1}	$1.196 * 10^{-3}$	$5.953 * 10^{-4}$	$2.970 * 10^{-4}$	$1.483 * 10^{-4}$	$7.412 * 10^{-5}$
2^{-2}	$1.302 * 10^{-3}$	$6.446 * 10^{-4}$	$3.207 * 10^{-4}$	$1.599 * 10^{-4}$	$7.987 * 10^{-5}$
2^{-3}	$1.380 * 10^{-3}$	$6.751 * 10^{-4}$	$3.340 * 10^{-4}$	$1.6610 * 10^{-4}$	$8.283 * 10^{-5}$
2^{-4}	$1.468 * 10^{-3}$	$7.024 * 10^{-4}$	$3.435 * 10^{-4}$	$1.699 * 10^{-4}$	$8.449 * 10^{-5}$
2^{-5}	$1.612 * 10^{-3}$	$7.411 * 10^{-4}$	$3.543 * 10^{-4}$	$1.733 * 10^{-4}$	$8.568 * 10^{-5}$
2^{-6}	$1.867 * 10^{-3}$	$8.104 * 10^{-4}$	$3.723 * 10^{-4}$	$1.779 * 10^{-4}$	$8.700 * 10^{-5}$
2^{-7}	$2.258 * 10^{-3}$	$9.366 * 10^{-4}$	$4.063 * 10^{-4}$	$1.866 * 10^{-4}$	$8.917 * 10^{-5}$
2^{-10}	$2.721 * 10^{-3}$	$1.361 * 10^{-3}$	$6.556 * 10^{-4}$	$2.836 * 10^{-4}$	$1.174 * 10^{-4}$
2^{-15}	$2.721 * 10^{-3}$	$1.364 * 10^{-3}$	$6.827 * 10^{-4}$	$3.416 * 10^{-4}$	$1.708 * 10^{-4}$
2^{-16}	$2.721 * 10^{-3}$	$1.364 * 10^{-3}$	$6.827 * 10^{-4}$	$3.416 * 10^{-4}$	$1.708 * 10^{-4}$
2^{-17}	$2.721 * 10^{-3}$	$1.364 * 10^{-3}$	$6.827 * 10^{-4}$	$3.416 * 10^{-4}$	$1.708 * 10^{-4}$
$D^{\Delta t}$	$2.721 * 10^{-3}$	$1.364 * 10^{-3}$	$6.827 * 10^{-4}$	$3.416 * 10^{-4}$	$1.708 * 10^{-4}$
$p^{\Delta t}$	0.996	0.998	0.999	1.000	$p^* = 0.996$
$C_{p^*}^{\Delta t}$	0.685	0.685	0.684	0.682	0.680

Table 5.3: Example 1. L^∞ norm of the numerical errors for different r and time steps, when $\mathcal{A}^l$ and $\mathbf{f}^l$ are as in (5.1) and (5.2), and $\varepsilon_1 = r/16$, $\varepsilon_2 = r/4$, $\varepsilon_3 = r$. The last column displays the numerical convergence order by fitting the log-log plot of the errors.

r \ Δt	1/128	1/256	1/512	1/1024	1/2048	order
2^{-1}	$1.590 * 10^{-4}$	$4.011 * 10^{-5}$	$9.941 * 10^{-6}$	$2.369 * 10^{-6}$	$4.738 * 10^{-7}$	2.09
2^{-2}	$3.047 * 10^{-4}$	$7.950 * 10^{-5}$	$1.988 * 10^{-5}$	$4.748 * 10^{-6}$	$9.504 * 10^{-7}$	2.07
2^{-3}	$5.261 * 10^{-4}$	$1.521 * 10^{-4}$	$3.934 * 10^{-5}$	$9.480 * 10^{-6}$	$1.902 * 10^{-6}$	2.02
2^{-4}	$7.520 * 10^{-4}$	$2.624 * 10^{-4}$	$7.517 * 10^{-5}$	$1.873 * 10^{-5}$	$3.792 * 10^{-6}$	1.91
2^{-5}	$9.316 * 10^{-4}$	$3.744 * 10^{-4}$	$1.293 * 10^{-4}$	$3.570 * 10^{-5}$	$7.473 * 10^{-6}$	1.73
2^{-6}	$1.116 * 10^{-3}$	$4.623 * 10^{-4}$	$1.835 * 10^{-4}$	$6.095 * 10^{-5}$	$1.412 * 10^{-5}$	1.55
2^{-7}	$1.352 * 10^{-3}$	$5.512 * 10^{-4}$	$2.242 * 10^{-4}$	$8.470 * 10^{-5}$	$2.342 * 10^{-5}$	1.44
2^{-10}	$2.502 * 10^{-3}$	$1.139 * 10^{-3}$	$4.593 * 10^{-4}$	$1.449 * 10^{-4}$	$4.091 * 10^{-5}$	1.48
2^{-15}	$2.643 * 10^{-3}$	$1.280 * 10^{-3}$	$5.978 * 10^{-4}$	$2.563 * 10^{-4}$	$8.543 * 10^{-5}$	1.22
2^{-16}	$2.643 * 10^{-3}$	$1.280 * 10^{-3}$	$5.978 * 10^{-4}$	$2.563 * 10^{-4}$	$8.543 * 10^{-5}$	1.22
2^{-17}	$2.643 * 10^{-3}$	$1.280 * 10^{-3}$	$5.978 * 10^{-4}$	$2.563 * 10^{-4}$	$8.543 * 10^{-5}$	1.22

For this example the results are given in Table 5.2, where it is seen that $p^* = 0.996$ and $C^*_{p^*} = 0.685$.

If we use

$$\mathcal{A}^l = \frac{1}{t_l - t_{l-1}}\Big(\int_{t_{l-1}}^{t_l} a_{i,j}(t)\,dt\Big)_{n\times n}, \tag{5.1}$$

$$\mathbf{f}^l = \frac{1}{t_l - t_{l-1}}\Big(\int_{t_{l-1}}^{t_l} f_1(t)\,dt, \int_{t_{l-1}}^{t_l} f_2(t)\,dt, \ldots, \int_{t_{l-1}}^{t_l} f_n(t)\,dt\Big)^T, \tag{5.2}$$

instead of $t_l^* = t_{l-1}$, the corresponding errors are displayed in Table 5.3. It can be seen that parameter-uniform first order convergence is achieved. However, second order convergence occurs when r is large, while when r decreases, only first order convergence is attained. This is different from the one dimensional neutron transport equation, where with cell averaging of the coefficients, parameter-uniform second order convergence is obtained [17]. The second order parameter-uniform convergence for the neutron transport equation is due to the specific scales of its coefficients. For the general form of problem (1.1)-(1.3), we can achieve only first order parameter-uniform convergence by using piecewise constant approximations of the coefficients. The computed parameter-uniform error parameters using the two-mesh method are given in Table 5.3.

Example 2. For a negative ε_i, a final value must be imposed in order to preserve the maximum principle. In this example, we solve a system with discontinuous coefficients such that for $t \in [0, 0.5]$,

$$\begin{aligned}
&-\varepsilon_1 u_1'(t) + (5 + e^{-t})u_1(t) - tu_2(t) - u_3(t) - u_4(t) = t,\\
&\varepsilon_2 u_2'(t) - u_1(t) + (4 + t^2)u_2(t) - u_3(t) - u_4(t) = 1,\\
&-\varepsilon_3 u_3' - u_1(t) - u_2(t) + 5u_3(t) - (1 + t)u_4(t) = 1 + t,\\
&\varepsilon_3 u_3' - u_1(t) - tu_2(t) - u_3(t) + 5u_4(t) = 1 - t^2,
\end{aligned}$$

and for $t \in [(0.5, 1]$,

$$\begin{aligned}
-\varepsilon_1 u_1'(t) + (4+e^{-t})u_1(t) - tu_2(t) - u_3(t) - u_4(t) &= 1,\\
\varepsilon_2 u_2'(t) - u_1(t) + (4+t^2)u_2(t) - u_3(t) - u_4(t) &= 1-t,\\
-\varepsilon_3 u_3' - u_1(t) - u_2(t) + (5+t^2)u_3(t) - (2+t)u_4(t) &= 1-t^2,\\
\varepsilon_3 u_3' - tu_1(t) - (1+t)u_2(t) - u_3(t) + (4+e^{-t})u_4(t) &= 1+t,
\end{aligned}$$

with the boundary conditions

$$u_1(1) = 0, \quad u_2(0) = 0, \quad u_3(1) = 0, \quad u_4(0) = 0.$$

This system exhibits initial, final and interface layers, when the $|\varepsilon_i|$ are small. Similarly to [23], let $\varepsilon_1 = r/64$, $\varepsilon_2 = r/16$, $\varepsilon_3 = r/4$, $\varepsilon_4 = r$. The numerical solutions of $u_1^{\varepsilon}(t)$ for different r are shown in Fig. 5.2. The results are consistent with first order parameter-uniform convergence, as expected from the theory.

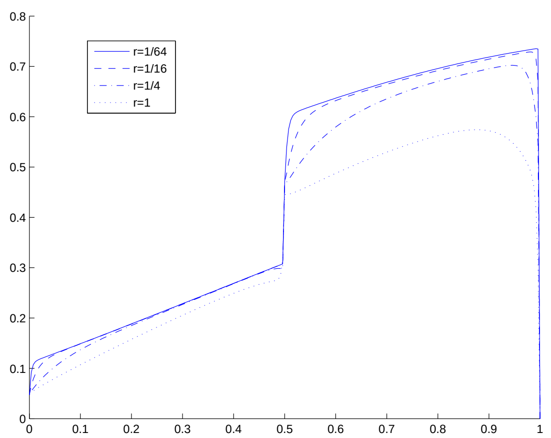


Fig. 5.2. Example 2. $u_1(t)$ for $r = 1, 1/4, 1/16, 1/64$.

In Table 5.4, we present the L^∞ norm of the numerical errors for different r and time steps, where we have used $t_l^* = t_{l-1}$ in (3.3) and (3.4). Uniform first order convergence can be observed.

To look at the uniform convergence order, we present in Table 5.5 the values of $D_\epsilon^{\Delta t}$, $D^{\Delta t}$, $p^{\Delta t}$ and $C_{p^*}^{\Delta t}$.

Example 3. In this example, we consider the following semi-discretization of the forward-backward parabolic problem

$$\operatorname{sign}(x_i)|x_i|^p \frac{\partial u_i(t)}{\partial t} = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} - u_i(t) + q_i(t), \quad 0 < t < T, \quad i = 1, 2, \cdots, N-1, \tag{5.3}$$

$$u_0(t) = f(t), \qquad u_N(t) = g(t), \quad 0 < t < T, \tag{5.4}$$

$$u_i(1) = \gamma(x), \quad -1 < x_i < 0, \qquad u_i(0) = s(x_i), \quad 0 < x_i < 2, \tag{5.5}$$

Table 5.4: Example 2. L^∞ norm of the numerical errors for different r and time steps, when $t_l^* = t_{l-1}$ in (3.3) and (3.4), $\varepsilon_1 = r/64$, $\varepsilon_2 = r/16$, $\varepsilon_3 = r/4$, $\varepsilon_4 = r$.

r \ Δt	1/128	1/256	1/512	1/1024	1/2048
1	$1.418 * 10^{-3}$	$6.920 * 10^{-4}$	$3.242 * 10^{-4}$	$1.392 * 10^{-4}$	$4.645 * 10^{-5}$
2^{-1}	$1.627 * 10^{-3}$	$7.813 * 10^{-4}$	$3.633 * 10^{-4}$	$1.554 * 10^{-4}$	$5.176 * 10^{-5}$
2^{-2}	$1.933 * 10^{-3}$	$9.222 * 10^{-4}$	$4.273 * 10^{-4}$	$1.824 * 10^{-4}$	$6.070 * 10^{-5}$
2^{-3}	$2.192 * 10^{-3}$	$1.036 * 10^{-3}$	$4.775 * 10^{-4}$	$2.033 * 10^{-4}$	$6.756 * 10^{-5}$
2^{-4}	$2.400 * 10^{-3}$	$1.118 * 10^{-3}$	$5.106 * 10^{-4}$	$2.163 * 10^{-4}$	$7.165 * 10^{-5}$
2^{-5}	$2.627 * 10^{-3}$	$1.194 * 10^{-3}$	$5.372 * 10^{-4}$	$2.254 * 10^{-4}$	$7.428 * 10^{-5}$
2^{-6}	$2.960 * 10^{-3}$	$1.293 * 10^{-3}$	$5.671 * 10^{-4}$	$2.345 * 10^{-4}$	$7.658 * 10^{-5}$
2^{-7}	$3.468 * 10^{-3}$	$1.451 * 10^{-3}$	$6.120 * 10^{-4}$	$2.469 * 10^{-4}$	$7.949 * 10^{-5}$
2^{-10}	$4.609 * 10^{-3}$	$2.166 * 10^{-3}$	$9.453 * 10^{-4}$	$3.592 * 10^{-4}$	$1.056 * 10^{-4}$
2^{-15}	$4.736 * 10^{-3}$	$2.293 * 10^{-3}$	$1.070 * 10^{-3}$	$4.587 * 10^{-4}$	$1.529 * 10^{-4}$
2^{-16}	$4.736 * 10^{-3}$	$2.293 * 10^{-3}$	$1.070 * 10^{-3}$	$4.587 * 10^{-4}$	$1.529 * 10^{-4}$
2^{-17}	$4.736 * 10^{-3}$	$2.293 * 10^{-4}$	$1.070 * 10^{-3}$	$4.587 * 10^{-4}$	$1.529 * 10^{-4}$

Table 5.5: Example 2. Values of D_ϵ, D, p, p, C_p^* and C_p^* with $\varepsilon_1 = r/64$, $\varepsilon_2 = r/16$, $\varepsilon_3 = r/4$, $\varepsilon_4 = r$, for various r and Δt. We find $p^* = \min_{\Delta t} p^{\Delta t} = 1.000$ and $C_{p^*}^* = \max_{\Delta t} C_{p^*}^{\Delta t} = 0.626$. Here $t_l^* = t_{l-1}$ in (3.3) and (3.4),

r \ Δt	1/128	1/256	1/512	1/1024	1/2048
1	$7.503 * 10^{-4}$	$3.678 * 10^{-4}$	$1.850 * 10^{-4}$	$9.277 * 10^{-5}$	$4.645 * 10^{-5}$
2^{-1}	$8.452 * 10^{-4}$	$4.180 * 10^{-4}$	$2.079 * 10^{-4}$	$1.037 * 10^{-4}$	$5.176 * 10^{-5}$
2^{-2}	$1.011 * 10^{-3}$	$4.950 * 10^{-4}$	$2.448 * 10^{-4}$	$1.217 * 10^{-4}$	$6.070 * 10^{-5}$
2^{-3}	$1.207 * 10^{-3}$	$5.588 * 10^{-4}$	$2.742 * 10^{-4}$	$1.358 * 10^{-4}$	$6.756 * 10^{-5}$
2^{-4}	$1.286 * 10^{-3}$	$6.140 * 10^{-4}$	$2.944 * 10^{-4}$	$1.446 * 10^{-4}$	$7.165 * 10^{-5}$
2^{-5}	$1.434 * 10^{-3}$	$6.569 * 10^{-4}$	$3.118 * 10^{-4}$	$1.511 * 10^{-4}$	$7.428 * 10^{-5}$
2^{-6}	$1.668 * 10^{-3}$	$7.256 * 10^{-4}$	$3.326 * 10^{-4}$	$1.578 * 10^{-4}$	$7.658 * 10^{-5}$
2^{-7}	$2.017 * 10^{-3}$	$8.394 * 10^{-4}$	$3.652 * 10^{-4}$	$1.674 * 10^{-4}$	$7.949 * 10^{-5}$
2^{-10}	$2.446 * 10^{-3}$	$1.221 * 10^{-3}$	$5.861 * 10^{-4}$	$2.536 * 10^{-4}$	$1.056 * 10^{-4}$
2^{-15}	$2.446 * 10^{-3}$	$1.223 * 10^{-3}$	$6.116 * 10^{-4}$	$3.058 * 10^{-4}$	$1.529 * 10^{-4}$
2^{-16}	$2.446 * 10^{-3}$	$1.223 * 10^{-3}$	$6.116 * 10^{-4}$	$3.058 * 10^{-4}$	$1.529 * 10^{-4}$
2^{-17}	$2.446 * 10^{-3}$	$1.223 * 10^{-3}$	$6.116 * 10^{-4}$	$3.058 * 10^{-4}$	$1.529 * 10^{-4}$
$D^{\Delta t}$	$2.446 * 10^{-3}$	$1.223 * 10^{-3}$	$6.116 * 10^{-4}$	$3.058 * 10^{-4}$	$1.529 * 10^{-4}$
$p^{\Delta t}$	1.000	1.000	1.000	1.000	$p^* = 1.000$
$C_{p^*}^{\Delta t}$	0.626	0.626	0.626	0.626	0.626

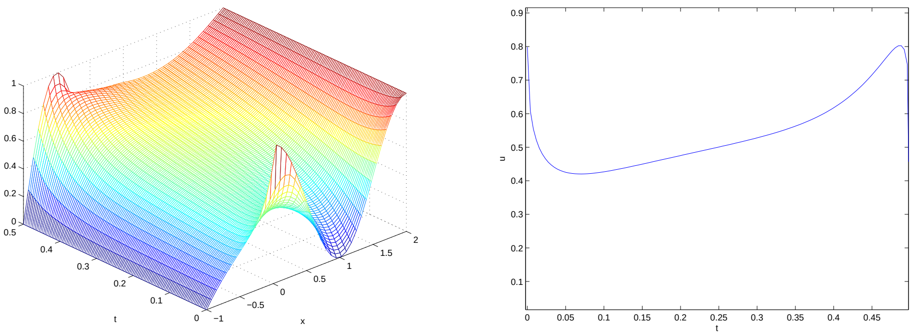
with $p > 0$ and $x_i = -1 + 3 * i/N \neq 0, \quad (1 \leq i \leq N-1)$.

In Fig. 5.3 we show the numerical results for

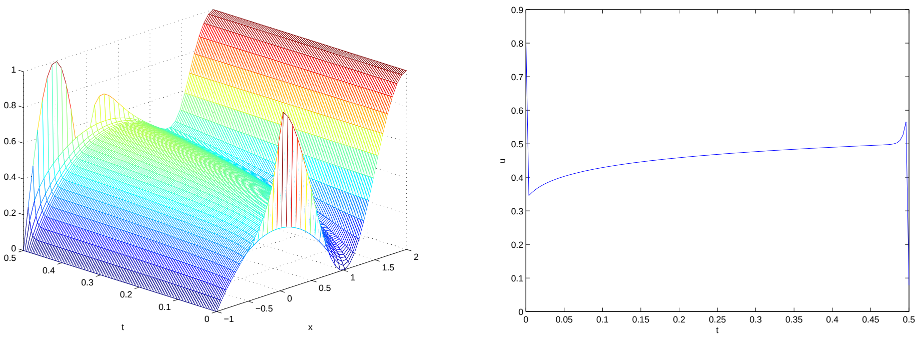
$$T = 0.5, \quad q_i(t) = 1, \quad \gamma(x) = -\sin(\pi x),$$

$$s(x) = \cos^2\left(\frac{\pi}{2}x\right), \quad f(t) = 0, \quad g(t) = 1.$$

Initial and final layers in time occur for both $p = 1$ and $p = 10$. The layers become more significant and obvious when p is large. We can capture the layers without resolving them (using a lot of nodes in the layer).



(a)



(b)

Fig. 5.3. Example 3. The numerical solution of the forward-backward parabolic problem. Left: $u_i(t)$ for $i = 1, \cdots, N$; Right: u_M where M satisfies $x_M < 0$ and $x_{M+1} > 0$. (a) $p = 1$; (b) $p = 10$. Here $\Delta x = 1/40$, $\Delta t = 1/256$.

6. Conclusion

A tailored finite point method for a multi-scale singularly perturbed system of linear ordinary differential equations is proposed in this paper. We can give either initial or final values for the ODE system as well as use discontinuous coefficients.

The tailored finite point method yields approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters. We prove a parameter-uniform error estimate in the maximum norm and verify our analytical results numerically. To show the performance of our proposed scheme, three numerical examples that exhibit initial and final layers are considered.

In particular, the third example shows the existence of initial and final layers of the solution of the forward-backward parabolic equation. To investigate and understand these layers will be the subject of our future work.

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