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THE REDUCED BASIS TECHNIQUE AS A COARSE SOLVER FOR PARAREAL IN TIME SIMULATIONS*

Liping He
Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, China
Email: lphe@sjtu.edu.cn

Abstract

In this paper, we extend the reduced basis methods for parameter dependent problems to the parareal in time algorithm introduced by Lions *et al.* [12] and solve a nonlinear evolutionary parabolic partial differential equation. The fine solver is based on the finite element method or spectral element method in space and a semi-implicit Runge-Kutta scheme in time. The coarse solver is based on a semi-implicit scheme in time and the reduced basis approximation in space. Offline-online procedures are developed, and it is proved that the computational complexity of the on-line stage depends only on the dimension of the reduced basis space (typically small). Parareal in time algorithms based on a multi-grids finite element method and a multi-degrees finite element method are also presented. Some numerical results are reported.

Mathematics subject classification: 52B10, 65D18, 68U05, 68U07.
Key words: Finite element and spectral element approximations, Multi-meshes and multi-degrees techniques, Reduced basis technique, Semi-implicit Runge-Kutta scheme, Offline-online procedure, Parareal in time algorithm.

1. Introduction

The parareal in time algorithm allows to use parallel computers for the approximation of the solution to ordinary or evolution partial differential equations by decomposing the time integration interval into time slabs and iterating on the resolution over each time slab to converge to the global solution. The iterations combine in a predictor/corrector way the use of a coarse propagator that is inexpensive and a precise solver (that is used only in parallel over each time slab, allocated to different processors); see, e.g., [2,3,17,18]. In many instances the iterative schemes provide an approximate solution as accurate as if the precise solver would be used over the complete time integration interval. One of the expensive parts of the solver is the resolution of the coarse solver since it is used sequentially over the complete time integration interval. Our goal is the development of numerical methods that permit the efficient evaluation of parareal in time simulation.

To achieve this goal we will pursue the reduced basis method. The reduced basis method was first introduced in the late 1970s for the nonlinear analysis of structures [1,19,20] and has subsequently been further investigated and developed more broadly; see, e.g., [4,5,9,21,22,24]. In the more recent past the reduced basis approach and in particular associated a posteriori error estimation procedures have been successfully developed for the PDEs with affine parameter or time dependence; see, e.g., [10,15,16,23,26]. Indeed, the reduced basis technique allows, from

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a few preliminary computations with a standard solver, to generate basis functions adapted to the further approximation of problems that depend on a parameter. This is a very high order approximation method, in the sense where, when the set of all solutions to the parameter dependent problem has a small width and with (much) less than 100 degrees of freedom, a very good approximation is available (the accuracy is about the one obtained with discretization). In more general cases where the dependency of the solutions in the parameter is not so regular, the number of degrees of freedom may become too large to get an acceptable accuracy. In this paper, we consider the extension of the reduced basis method to define a coarse and very cheap propagator that allows to get the full efficiency in a parareal context.

Many numerical methods are considered to define the coarse propogator in the literature [17]: the most well-known ones are the usual coarse mesh of the finite element method (FEM), the spectral approximation space based on the polynomial of lower degree, and a coarser model based on simpler physics. The success of these numerous experiments not only richen the idea of parareal in time algorithm, but also motivate the need for further studies in this direction. The main contributions here are as follows: (i) we construct a coarse propagator based on a semi-implicit scheme in time and the reduced basis approximation in space, and prove that the computational complexity of the on-line stage of the procedure scales only with the dimension of the reduced basis space (this fact means that good accuracy is obtained even for very few basis functions, and thus the computational cost of the coarse solver is typically very small); (ii) we propose a fine propagator based on the FEM or spectral element method (SEM) in space and a semi-implicit Runge-Kutta (RK) scheme in time; (iii) the parareal in time algorithm based on a multi-degrees FEM in space and the semi-implicit RK scheme in time is considered.

This paper is organized as follows: Section 2 describes the basic algorithm for a model equation. Section 3 introduces the necessary notations and the initial-boundary problem which is considered in this paper and proposes two types of fine approximated propagators based on the FEM and SEM in space and semi-implicit RK scheme in time. Section 4 introduces the coarse approximated propagator based on the reduced basis method. Section 5 gives the parareal in time algorithms based on the multi-grids FEM and the multi-degrees FEM. Some numerical results are reported in Section 6, and finally we give some conclusions in Section 7.

2. Basic Algorithm on a Model Equation

Consider the following time dependent problem

$$\frac{\partial u}{\partial t} + Lu = 0, \quad u(0) = u^0, \tag{2.1}$$

where, for the sake of simplicity, the operator L does not depend on time. We introduce the propagator S such that $S_\tau(v)$ is the solution, at time τ of the problem

$$\frac{\partial u}{\partial t} + Lu = 0, \quad u(0) = v. \tag{2.2}$$

Due to time invariance, it is well-known that

$$S_\tau = S_{\tau - t} \circ S_t, \qquad \forall t < \tau. \tag{2.3}$$

Let $0 = T_0 < T_1 < \cdots < T_n < \cdots < T_M = T$ be special times at which we are interested to consider snapshots of the solution $u(T_n)$. Then we obtain from (2.2) and (2.3) that

$$u(T_{n+1}) = S_{T_{n+1}}(u^0) = S_{T_{n+1} - T_n}(u_{T_n}).$$

In most cases S is not achievable but only approximations based on time discretization and the use of Euler or more involved schemes. For instance, we can introduce a fine and precise approximated propagator $\mathcal{F}$ defined through the Runge-Kutta scheme of (2.1), which reads

$$\frac{u^{m+1}-u^m}{\delta t}+\frac{1}{2}(Lu^m+Lu^{m+1})=0 \tag{2.4}$$

for any time $T=M\delta t$ the approximated propagator $\mathcal{F}_T$ involves the iterative resolution of M problems as above. Similarly as for the continuous solution, we have the approximations u_n of $u(T_n)$ given by

$$\mathrm{u}_{n+1}=\mathcal{F}_{T_{n+1}}(u^0)=\mathcal{F}_{T_{n+1}-T_n}(\mathrm{u}_n).$$

Assuming, for the sake of simplicity, that $T_{n+1}-T_n$ is constant ($\Delta T\gg\delta t$), then this reads

$$\mathrm{u}_{n+1}=\mathcal{F}_{\Delta T}(\mathrm{u}_n), \tag{2.5}$$

where it appears that the approximated solution process is sequential, which, prevents it from a parallelization.

In what follows Lions *et al.* [12] proposed an algorithm $\mathrm{u}_n^k\longrightarrow\mathrm{u}_n$ as k goes to infinity. For this we assume that another propagator $\mathcal{G}$ is achievable. It is assumed to be cheap but inaccurate. One can think about $\mathcal{G}$ based on the same time discretization as $\mathcal{F}$ but with larger time step Δt. Other possibility may be offered as e.g. $\mathcal{F}$ carries all the physics of the phenomenon but $\mathcal{G}$ is based on a simplified physics. Then, the iterative process is

$$\mathrm{u}_{n+1}^{k+1}=\mathcal{G}_{\Delta T}(\mathrm{u}_n^{k+1})+\mathcal{F}_{\Delta T}(\mathrm{u}_n^k)-\mathcal{G}_{\Delta T}(\mathrm{u}_n^k). \tag{2.6}$$

This iterative process provides a converging sequence, in the sense that

$$\text{if}\quad |S_{\Delta T}-\mathcal{F}_{\Delta T}|\simeq\delta t^2\quad\text{and if}\quad|\mathcal{G}_{\Delta T}-\mathcal{F}_{\Delta T}|\simeq\varepsilon\Delta T, \tag{2.7}$$

after k iterations the error between u_n^k and $u(T_n)$ is $\varepsilon^k+\delta t^2$.

3. Fine Approximated Propagator

Let $T>0$, $\mu>0$ be the kinetic viscosity, Δ be the Laplacian, and $\partial\Omega$ be the boundary of Ω. $f(t,x,y)$ and $u_0(x,y)$ describe the source term and the initial state. Denote by $\phi(u)=u^3$ the nonlinear term. Then the nonlinear evolutionary parabolic partial differential equation to be considered (see [10,17]) is of the form

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x}-\mu\Delta u+\phi(u)=f, & \text{in} \quad \Omega\times(0,T],\\ u=0, & \text{on}\quad \partial\Omega\times(0,T],\\ u(x,y,0)=u_0(x,y), & \text{in}\quad \Omega\cup\partial\Omega.\end{cases} \tag{3.1}$$

Throughout the paper we use Sobolev space $H^r(\Omega)$ and $H_0^r(\Omega)$. For simplicity, let $L^2(\Omega)=H^0(\Omega)$. The inner product, the semi-norm and the norm of $H^r(\Omega)$, $r\geq 0$, are denoted by $(\cdot,\cdot)_r$, $|\cdot|_r$, $\|\cdot\|_r$ respectively. If $r=0$, then the index r is omitted. We recall that the usual semi-norm $|\cdot|_r$ is equivalent to the norm $\|\cdot\|_r$ in $H_0^r(\Omega)$. Further let $H^{-r}(\Omega)$ be the dual space of $H_0^r(\Omega)$, and $\langle\cdot,\cdot\rangle_{L(H^{-r},H_0^r)}$ be the duality parting between $H^{-r}(\Omega)$ and $H_0^r(\Omega)$. Let $\frac{\partial u}{\partial x}=\partial_x u$.

We show the weak formulation of (3.1). For given functions $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, the weak solution of (3.1) is to find a function $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} \left(\dfrac{\partial}{\partial t}u, v\right) + (\partial_x u, v) + \mu(\nabla u, \nabla v) + (\phi(u), v) = \langle f, v\rangle_{L(H^{-r}, H_0^r)}, \quad \forall v \in H_0^1(\Omega), \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{3.2}$$

It can be proved that if $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, then (3.2) has a unique solution $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

3.1. Semi-implicit time discretization schemes

We first consider a time-discrete framework associated with the time interval $I =]0,T]$. We divide $\bar{I} = [0,T]$ into $M_{\delta t}$ subintervals of equal length $\delta t = T/M_{\delta t}$ and define $t_{\delta t}^m = m\delta t$, $0 \le m \le M_{\delta t} \equiv T/\delta t$, and $I_{\delta t} \equiv \{1,2,\cdots,M_{\delta t}\}$. Also, we divide $\bar{I}$ into $M_{\Delta t}$ subintervals of equal length $\Delta t = T/M_{\Delta t}$ and define $t_{\Delta t}^m = m\Delta t$, $0 \le m \le M_{\Delta t} \equiv T/\Delta t$, and $I_{\Delta t} \equiv \{1,2,\cdots,M_{\Delta t}\}$. Given a time step τ, set $u^m(x,y) = u(x,y,m\tau)$, also denoted by u^m for simplicity. Let $u^m \in H_0^1(\Omega)$ be the approximation to the solution of (3.2) at time $t_\tau^m = m\tau$. Denoted by u_{pre}^{m+1} the predicted value of u^{m+1}. The average operators A_τ and A_τ^{pre}, and the difference operators ∂_τ and ∂_τ^{pre} are defined as

$$A_\tau u^m = \frac{1}{2}\Big(u(x,y,t_\tau^m) + u(x,y,t_\tau^{m+1})\Big), \quad A_\tau^{pre} u^m = \frac{1}{2}\Big(u(x,y,t_\tau^m) + u_{pre}(x,y,t_\tau^{m+1})\Big),$$

$$\partial_\tau u^m = \frac{1}{\tau}\Big(u(x,y,t_\tau^{m+1}) - u(x,y,t_\tau^m)\Big), \quad \partial_\tau^{pre} u^m = \frac{1}{\tau}\Big(u_{pre}(x,y,t_\tau^{m+1}) - u(x,y,t_\tau^m)\Big).$$

In this paper the fine time discretization scheme of (3.2) is defined by the semi-implicit RK scheme (see [11]): find $u^{m+1} \in H_0^1(\Omega)$ such that

$$\begin{cases} (\partial_{\delta t}^{pre} u^m, v) + (\partial_x u^m, v) + \mu(\nabla A_{\delta t}^{pre} u^m, \nabla v) + (\phi(u^m), v) = (f^m, v), \quad \forall v \in H_0^1(\Omega), \\ (\partial_{\delta t} u^m, v) + (\partial_x A_{\delta t}^{pre} u^m, v) + \mu(\nabla A_{\delta t} u^m, \nabla v) + (A_{\delta t}^{pre}\phi(u^m), v) \\ \qquad\qquad = (A_{\delta t} f^m, v), \quad \forall v \in H_0^1(\Omega), \quad \forall m \in I_{\delta t}, \\ u^0 = u_0, \end{cases} \tag{3.3}$$

and the coarse time discretization scheme of (3.2) is defined by a semi-implicit scheme: find $u^{p+1} \in H_0^1(\Omega)$ such that

$$\begin{cases} (\partial_{\Delta t} u^p, v) + (\partial_x u^p, v) + \mu(\nabla A_{\Delta t} u^p, \nabla v) + (\phi(u^p), v) = (f^p, v), \\ \qquad\qquad \forall v \in H_0^1(\Omega), \quad \forall p \in I_{\Delta t}, \\ u^0 = u_0. \end{cases} \tag{3.4}$$

It is well-known that the time step δt and Δt are dependent on the spacial discretization of problem (3.2).

3.2. Spectral element approximation

The SEM is based on a decomposition of the global domain, $\bar{\Omega} = \Omega \cup \partial\Omega$, into E nonoverlapping subdomains, Ω^e, $e = 1,2,\cdots,E$. Each subdomain Ω^e is a deformed quadrilateral in R^2

and can be considered as a mapping of a reference domain $\hat{\Omega} = [-1, 1]^2$. We write $\Omega^e = \Phi_e(\hat{\Omega})$, where Φ_e is an one-to-one mapping. In terms of the reference variables, we have that

$$\int_{\Omega^e} uvd\Omega = \int_{\hat{\Omega}} (u \circ \Phi_e)(v \circ \Phi_e)|J_e|d\hat{\Omega}, \tag{3.5}$$

$$\int_{\Omega^e} \nabla u \cdot \nabla v d\Omega = \int_{\hat{\Omega}} \mathcal{J}_e^{-T} \hat{\nabla}(u \circ \Phi_e) \cdot \mathcal{J}_e^{-T} \hat{\nabla}(v \circ \Phi_e)|J_e|d\hat{\Omega}, \tag{3.6}$$

where $\mathcal{J}_e$ is the Jacobian of Φ_e,

$$\mathcal{J}_e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix},$$

and $|J_e|$ denotes the determinant of $\mathcal{J}_e$.

We now consider a discretization of (3.5) and (3.6). Let $P_N(\hat{\Omega})$ be the space of all functions which are polynomials of degree less than or equal to N in each spatial direction on the reference domain $\hat{\Omega}$. The discrete space of piecewise continuous functions that map to the polynomials in the reference domain $\hat{\Omega}$ is then taken to be

$$P_{N,E}(\Omega) = \left\{ v \in H_0^1(\Omega) : v \circ \Phi_e \in P_N(\hat{\Omega}), \;\; e = 1, 2, \cdots, E \right\}.$$

The basis of $P_N(\hat{\Omega})$ is conveniently expressed in terms of the reference variables ξ and η. As a basis for $P_N(\hat{\Omega})$ we use a nodal basis through the tensor-product Gauss-Lobatto Legendre (GLL) points; see,e.g., [8,13,14,25]. Specifically, we write $u_N \in P_{N,E}(\Omega)$ as

$$u_N \circ \Phi_e(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} u_{i,j}^e \pi_i(\xi) \pi_j(\eta),$$

where $u_{i,j}^e$ represents nodal values for the element e and $\pi_i(\xi)$ refers to a one-dimensional N-th order Lagrangian interpolant through the GLL points ξ_m, $m = 0, 1, \cdots, N$; here, $\pi_i(\xi_m)\pi_j(\xi_n) = \delta_{im}\delta_{jn}$ for a given point (ξ_m, ξ_n) in the underlying tensor-product GLL grid. Within the framework of multielement discretizations, the scalar product and the bilinear form are defined by

$$(u_N, v_N)_N = \sum_{e=1}^{E} \int_{\hat{\Omega}} (u_N \circ \Phi_e)(v_N \circ \Phi_e)|J_e|d\hat{\Omega}, \qquad \forall u_N, v_N \in P_{N,E}(\Omega),$$

$$(\nabla u_N, \nabla v_N)_N = \sum_{e=1}^{E} \int_{\hat{\Omega}} \mathcal{J}_e^{-T} \hat{\nabla}(u_N \circ \Phi_e) \cdot \mathcal{J}_e^{-T} \hat{\nabla}(v_N \circ \Phi_e)|J_e|d\hat{\Omega}, \quad \forall u_N, v_N \in P_{N,E}(\Omega).$$

In the SEM, GLL quadrature is used to the evaluation of the integrals, resulting in local forms similar to the spectral method. The other terms of (3.2) can be approximated similarly.

We denote by $V_N(\Omega)$ the subspace $P_{N,E}(\Omega) \cap H^1(\Omega)$ and by $\partial\Omega_D$ the subset of $\partial\Omega$ on which homogeneous Dirichlet boundary conditions are enforced. The underlying approximation space in the SEM is defined as

$$V_{N,0}(\Omega) = \left\{ v \in V_N(\Omega) : v|_{\partial\Omega_D} = 0 \right\}.$$

We define the orthogonal projection operator $\Pi_{1,N} : H_0^1(\Omega) \mapsto V_{N,0}(\Omega)$ such that

$$(\nabla(u - \Pi_{1,N}u), \nabla v_N)_N = 0, \quad \forall v_N \in V_{N,0}(\Omega).$$

We now consider the discretization only in space. The semi-discrete spectral element approximation of (3.2) is a function $u_N(x,y,t) \in V_{N,0}(\Omega)$ such that

$$\begin{cases} \left(\dfrac{\partial}{\partial t}u_N, v_N\right)_N + (\partial_x u_N, v_N)_N + \mu(\nabla u_N, \nabla v_N)_N + (\phi(u_N), v_N)_N \\ \qquad = (f, v_N)_N, \ \ \forall v_N \in V_{N,0}(\Omega), \\ u_N(x,y,0) = \Pi_{1,N} u_0(x,y). \end{cases} \tag{3.7}$$

We describe the fully discrete scheme. We divide $\bar{I}$ into $M_{\rm se}$ subintervals of equal length $\delta t_{\rm se} = T/M_{\rm se}$ and define $I_{\rm se} \equiv \{1, 2, \cdots, M_{\rm se}\}$. Let $u_N^m \in V_{N,0}(\Omega)$ be the approximation to the solution of (3.2) at time $t^m = m\delta t_{\rm se}$. Then the semi-implicit RK spectral element approximation for (3.2) is of the form: find $u_N^{m+1} \in V_{N,0}(\Omega)$ such that

$$\begin{cases} (\partial^{pre}_{\delta t_{\rm se}} u_N^m, v_N)_N + (\partial_x u_N^m, v_N)_N + \mu(\nabla A^{pre}_{\delta t_{\rm se}} u_N^m, \nabla v_N)_N \\ \qquad +(\phi(u_N^m), v_N)_N = (f^m, v_N)_N, \qquad \forall v_N \in V_{N,0}(\Omega), \\ (\partial_{\delta t_{\rm se}} u_N^m, v_N)_N + (\partial_x A^{pre}_{\delta t_{\rm se}} u_N^m, v_N)_N + \mu(\nabla A_{\delta t_{\rm se}} u_N^m, \nabla v_N)_N \\ \qquad +(A^{pre}_{\delta t_{\rm se}} \phi(u_N^m), v_N)_N = (A_{\delta t_{\rm se}} f^m, v_N)_N, \ \ \forall v_N \in V_{N,0}(\Omega), \forall m \in I_{\rm se}, \\ u_N^0 = \Pi_{1,N} u_0. \end{cases} \tag{3.8}$$

Obviously the approximation solution on the initial level is well defined. Now assume that the numerical solution on the mth level has been calculated. Let

$$a_N(u_N, v_N) = (u_N, v_N)_N + \frac{1}{2}\mu\delta t_{\rm se}(\nabla u_N, \nabla v_N)_N, \ \ \forall u_N, v_N \in V_{N,0}(\Omega)$$

be the prediction-correction operator. Clearly, $a_N(u_N, v_N)$ is a bilinear continuous and coercive form on $V_{N,0}(\Omega) \times V_{N,0}(\Omega)$. Hence by Lax-Milgram theorem, the numerical solution on the $m+1$th level is determined uniquely. So this scheme has a unique solution on each level as long as $f \in C(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. Moreover, the above scheme is the second-order in time and high order in space (see [11]).

3.3. Finite element approximation

Let us consider a family of regular triangulation $\{T_h\}$ in $\bar{\Omega}$. Denote by Ω_h the triangulated domain of Ω, and by h_e the diameter of the closed triangulation element e. The mesh parameter is defined as $h = \max_{e \in T_h}\{h_e\}$. Denote by $P_s(e), e \in T_h$, the spaces which contain polynomials of degree s. Define

$$V_h^s(\Omega_h) = \left\{ v_h \in H^1(\Omega_h) : v_h|_e \in P_s(e), \forall e \in T_h \right\}.$$

Thus, a function $v_h \in V_h^1(\Omega_h)$ (i) is such that each restriction $v_h|_e$ is in the space $P_1(e)$ for each $e \in T_h$ and (ii) is completely determined by its values at all the vertices of the triangulation. Likewise, a function of $V_h^2(\Omega_h)$ (i) is in the space $P_2(e)$ for each $e \in T_h$, and (ii) is completely determined by its values at all the vertices and all the mid-points of the edges of the triangulation. Let $s = 1$ or 2. The finite element approximation space is defined as

$$V_{h,0}^s(\Omega_h) = \left\{ v_h \in V_h^s(\Omega_h) : v_h|_{\partial\Omega_h} = 0 \right\}.$$

Denote the node set of T_h by $\{p_j\}_{j=1}^{J_h^s}$, in which p_j $(1 \le j \le K_h^s)$ is the inner node of Ω_h, and p_k $(K_h^s + 1 \le k \le J_h^s)$ is the boundary node on $\partial\Omega_h$. It is obvious that

$$\dim V_h^s(\Omega_h) = J_h^s, \ \ \dim V_{h,0}^s(\Omega_h) = K_h^s.$$

Define the V_h^s-interpolation operator Π_h^s: $\mathrm{dom}\Pi_h^s \mapsto V_h^s(\Omega)$ such that

$$\psi_j(\Pi_h^s v) = \psi_j(v), \quad \forall v \in \mathrm{dom}\Pi_h^s, \quad j = 1, 2, \cdots, J_h^s.$$

Thus, if $v \in \mathrm{dom}\Pi_h^s$ and $v|_{\partial\Omega_h} = 0$, then we have $\Pi_h^s v \in V_{h,0}^s(\Omega)$ (see [6]). Define

$$\begin{aligned}(u_h, v_h)_h &= \sum_{e\in T_h} \int_e u_h(\mathbf{x}) v_h(\mathbf{x}) d\mathbf{x}, (\nabla u_h, \nabla v_h)_h \\ &= \sum_{e\in T_h} \int_e \nabla u_h(\mathbf{x}) \nabla v_h(\mathbf{x}) d\mathbf{x}, \quad \forall u_h, v_h \in V_h^s(\Omega). \end{aligned} \tag{3.9}$$

Denote by $\hat{e}$ the reference finite element, and let $F_e : \hat{\mathbf{x}} \in \hat{e} \mapsto \mathbf{x} = F_e(\hat{\mathbf{x}}) = B_e\hat{\mathbf{x}} + b_e \in e$ be the invertible affine mapping which maps $\hat{e}$ onto e. Then

$$\int_e u_h(\mathbf{x}) v_h(\mathbf{x}) \mathrm{d}\mathbf{x} = \det(B_e) \int_{\hat{e}} u_h(F_e(\hat{\mathbf{x}})) v_h(F_e(\hat{\mathbf{x}})) \mathrm{d}\hat{\mathbf{x}}, \quad \forall u_h, v_h \in V_h^s(\Omega).$$

Consequently, for each s, we need only a numerical quadrature scheme over the reference finite element to approximating the integrals which appear in (3.9).

The semi-discrete finite element approximation of (3.2) is a function $u_h \in V_{h,0}^s(\Omega)$, $s = 1, 2$, such that

$$\begin{cases} \left(\dfrac{\partial}{\partial t} u_h, v_h\right)_h + (\partial_x u_h, v_h)_h + \mu(\nabla u_h, \nabla v_h)_h + (\phi(u_h), v_h)_h \\ \qquad = (f, v_h)_h, \quad \forall v_h \in V_{h,0}^s(\Omega), \\ u_h(x, y, 0) = \Pi_h^s u_0(x, y). \end{cases} \tag{3.10}$$

We divide $\bar{I} = [0, T]$ into M_{fe} subintervals of equal length $\delta t_{\mathrm{fe}} = T/M_{\mathrm{fe}}$ and define $I_{\mathrm{fe}} \equiv \{1, 2, \cdots, M_{\mathrm{fe}}\}$. Let $u_h^m \in V_{h,0}(\Omega)$ be the approximation to the solution of (3.2) at time $t^m = m\delta t_{\mathrm{fe}}$. Denoted by $u_{h,pre}^{m+1}$ the predicted value of u_h^{m+1}. Then the semi-implicit RK finite element approximation for (3.2) is of the form: find $u_h^{m+1} \in V_{h,0}^s(\Omega)$, $s = 1, 2$, such that

$$\begin{cases} (\partial_{\delta t_{\mathrm{fe}}}^{pre} u_h^m, v_h)_h + (\partial_x u_h^m, v_h)_h + \mu(\nabla A_{\delta t_{\mathrm{fe}}}^{pre} u_h^m, \nabla v_h)_h \\ \qquad +(\phi(u_h^m), v_h)_h = (f^m, v_h)_h, \qquad \forall v_h \in V_{h,0}^s(\Omega), \\ (\partial_{\delta t_{\mathrm{fe}}} u_h^m, v_h)_h + (\partial_x A_{\delta t_{\mathrm{fe}}}^{pre} u_h^m, v_h)_h + \mu(\nabla A_{\delta t_{\mathrm{fe}}} u_h^m, \nabla v_h)_h \\ \qquad +(A_{\delta t_{\mathrm{fe}}}^{pre} \phi(u_h^m), v_h)_h = (A_{\delta t_{\mathrm{fe}}} f^m, v_h)_h, \ \ \forall v_h \in V_{h,0}^s(\Omega), \forall m \in I_{\mathrm{fe}}, \\ u_h^0 = \Pi_h^s u_0. \end{cases} \tag{3.11}$$

Similarly, the prediction-correction operator is now defined as

$$a(u_h, v_h) = (u_h, v_h)_h + \frac{1}{2}\mu\delta t_{\mathrm{fe}}(\nabla u_h, \nabla v_h)_h, \quad \forall u_h, v_h \in V_{h,0}^s(\Omega).$$

From Lax-Milgram theorem, we can show that this scheme has a unique solution on each time level.

4. Coarse Approximated Propagator

We first introduce the nested sample sets $S_{N_r}^{\mathrm{se}} = \{m_1\delta t_{\mathrm{se}}, m_2\delta t_{\mathrm{se}}, \cdots, m_{N_r}\delta t_{\mathrm{se}}\}$, $1 \le N_r \le N_{\mathrm{se,max}}$, and $S_{N_s}^{\mathrm{fe}} = \{n_1\delta t_{\mathrm{fe}}, n_2\delta t_{\mathrm{fe}}, \cdots, n_{N_s}\delta t_{\mathrm{fe}}\}$, $1 \le N_s \le N_{\mathrm{fe,max}}$; note that the samples must

reside in the discrete time space. Here, N_r and N_s are the dimensions of the reduced basis spaces for the time parameter. We then define the reduced basis spaces

$$W_{N_r}^{\rm se} = {\rm span}\left\{\zeta_1^{\rm se} = u_N^{m_1}, \zeta_2^{\rm se} = u_N^{m_2}, \cdots, \zeta_{N_r}^{\rm se} = u_N^{m_{N_r}}\right\}, \tag{4.1}$$

and

$$W_{N_s}^{\rm fe} = {\rm span}\left\{\zeta_1^{\rm fe} = u_h^{n_1}, \zeta_2^{\rm fe} = u_h^{n_2}, \cdots, \zeta_{N_s}^{\rm fe} = u_h^{n_{N_s}}\right\}, \tag{4.2}$$

where $u_N^{m_k}$ is the solution of (3.8) at time $t = m_k \delta t_{\rm se}$ and $u_h^{n_k}$ is the solution of (3.11) at time $t = n_k \delta t_{\rm fe}$ with $s = 2$.

We divide $\bar{I}$ into $\mathcal{M}_{\rm se}$ subintervals of equal length $\Delta t_{\rm se} = T/\mathcal{M}_{\rm se}$ and define $t^p = p\Delta t_{\rm se}$, $0 \le p \le \mathcal{M}_{\rm se} \equiv T/\Delta t_{\rm se}$, and $\mathcal{I}_{\rm se} \equiv \{1, 2, \cdots, \mathcal{M}_{\rm se}\}$. Also, we divide $\bar{I}$ into $\mathcal{M}_{\rm fe}$ subintervals of equal length $\Delta t_{\rm fe} = T/\mathcal{M}_{\rm fe}$ and define $t^q = q\Delta t_{\rm fe}$, $0 \le q \le \mathcal{M}_{\rm fe} \equiv T/\Delta t_{\rm fe}$, and $\mathcal{I}_{\rm fe} \equiv \{1, 2, \cdots, \mathcal{M}_{\rm fe}\}$.

4.1. Semi-implicity spectral element reduced basis approximation

The spectral element reduced basis approximation $u_{N_r}^{p+1}(x, y)$ to $u(x, y, t^{p+1})$ is defined by a standard Galerkin projections: find $u_{N_r}^{p+1} \in W_{N_r}^{\rm se}$ such that

$$\begin{cases} (\partial_{\Delta t_{\rm se}} u_{N_r}^p, v_{N_r})_N + (\partial_x u_{N_r}^p, v_{N_r})_N + \mu(\nabla A_{\Delta t_{\rm se}} u_{N_r}^p, \nabla v_{N_r})_N \\ \qquad\qquad + (\phi(u_{N_r}^p), v_{N_r})_N = (f^p, v_{N_r})_N, \quad \forall v_{N_r} \in W_{N_r}^{\rm se}, \forall p \in \mathcal{I}_{\rm se}, \\ u_{N_r}^0 = \Pi_{1,N} u_0. \end{cases} \tag{4.3}$$

To ensure stability of the reduced basis solution, all basis functions are orthogonalized, but we keep the basis functions representing different time separated. For the spectral element reduced basis functions, the orthogonalization is done with respect to the discrete inner product $(\cdot, \cdot)_N + \frac{1}{2}\mu\Delta t_{\rm se}(\nabla\cdot, \nabla\cdot)_N$. Let $\{\varphi_i\}_{i=1,2,\cdots,N_r}$ be the orthogonal basis functions of $W_{N_r}^{\rm se}$ and $u_{N_r}^p = \sum_{k=1}^{N_r} \hat{u}_k^p \varphi_k$. Then we have

$$\phi(u_{N_r}^p) = \sum_{k=1}^{N_r} (\hat{u}_k^p)^3 \varphi_k^3 + 3 \sum_{\substack{k,l=1 \\ k\neq l}}^{N_r} (\hat{u}_k^p)^2 \hat{u}_l^p \varphi_k^2 \varphi_l + 6 \sum_{\substack{i,k,l=1 \\ i<k<l}}^{N_r} \hat{u}_i^p \hat{u}_k^p \hat{u}_l^p \varphi_i \varphi_k \varphi_l,$$

which gives us

$$\begin{aligned} (\phi(u_{N_r}^p), \varphi_j)_N = &\sum_{k=1}^{N_r} (\hat{u}_k^p)^3 (\varphi_k^3, \varphi_j)_N \\ &+ 3 \sum_{\substack{k,l=1 \\ k\neq l}}^{N_r} (\hat{u}_k^p)^2 \hat{u}_l^p (\varphi_k^2 \varphi_l, \varphi_j)_N + 6 \sum_{\substack{i,k,l=1 \\ i<k<l}}^{N_r} \hat{u}_i^p \hat{u}_k^p \hat{u}_l^p (\varphi_i \varphi_k \varphi_l, \varphi_j)_N. \end{aligned}$$

Note that

$$(\varphi_k, \varphi_j)_N + \frac{1}{2}\mu\Delta t_{\rm se}(\nabla\varphi_k, \nabla\varphi_j)_N = \delta_{k,j}.$$

Then, we find that

$$\begin{aligned} \hat{u}_j^{p+1} = &2(u_{N_r}^p, \varphi_j)_N - \hat{u}_j^p - \Delta t_{\rm se}(\partial_x u_{N_r}^p, \varphi_j)_N \\ &- \Delta t_{\rm se}(\phi(u_{N_r}^p), \varphi_j)_N + \Delta t_{\rm se}(f^p, \varphi_j)_N, \quad j = 1, 2, \cdots, N_r. \end{aligned} \tag{4.4}$$

Off-line Stage

In the offline stage, all of the following precomputing are performed only once.

1. Calculate $\zeta_i^{\rm se} = u_N^{m_i}$, $i = 1, \cdots, N_r$, to form $W_{N_r}^{\rm se}$ as in (4.1).
2. Form the orthogonal basis functions $\{\varphi_i\}_{i=1,\cdots,N_r}$ of $W_{N_r}^{\rm se}$.
3. Evaluate and store $B_{i,j}^{(0)} = 2(\varphi_i, \varphi_j)_N$, $B_{i,j}^{(1)} = \Delta t_{\rm se}(\partial_x \varphi_i, \varphi_j)_N$, $i, j = 1, \cdots, N_r$.
4. Calculate and store $B_{i,j}^{(2)} = \Delta t_{\rm se}(\varphi_i^3, \varphi_j)_N$, $C_{i,j,k} = 3\Delta t_{\rm se}(\varphi_i^2\varphi_j, \varphi_k)_N$ and $D_{i,j,k,l} = 6\Delta t_{\rm se}(\varphi_i\varphi_j\varphi_k, \varphi_l)_N$.
5. Compute and store the linear functions $F_j^{(p)} = \Delta t_{\rm se}(f^p, \varphi_j)_N$, $j = 1, \cdots, N_r$, $\forall p \in P^{\rm se}$.

On-line Stage

Let $A_j^{(0,p)} = 2(u_{N_r}^p, \varphi_j)_N$, $A_j^{(1,p)} = \Delta t_{\rm se}(\partial_x u_{N_r}^p, \varphi_j)_N$, $A_j^{(2,p)} = \Delta t_{\rm se}(\phi(u_{N_r}^p), \varphi_j)_N$.

1. Compute $A_j^{(0,p)} = \sum_{k=1}^{N_r} \hat{u}_k^p B_{k,j}^{(0)}$, $A_j^{(1,p)} = \sum_{k=1}^{N_r} \hat{u}_k^p B_{k,j}^{(1)}$, $j = 1, 2, \cdots, N_r$.
2. Evaluate the inner product of the nonlinear term as

$$A_j^{(2,p)} = \sum_{k=1}^{N_r} (\hat{u}_k^p)^3 B_{k,j}^{(2)} + \sum_{\substack{k,l=1 \\ k\neq l}}^{N_r} (\hat{u}_k^p)^2 \hat{u}_l^p C_{k,l,j} + \sum_{\substack{i,k,l=1 \\ i<k<l}}^{N_r} \hat{u}_i^p \hat{u}_k^p \hat{u}_l^p D_{i,k,l,j}, \ j = 1, 2, \cdots, N_r.$$

3. Calculate $\hat{u}_j^{p+1} = A_j^{(0,p)} - \hat{u}_j^p - A_j^{(1,p)} - A_j^{(2,p)} + F_j^{(p)}$, $j = 1, 2, \cdots, N_r$.

This stage requires $2N_r$ evaluations of $A_j^{(0,p)}$ and $A_j^{(1,p)}$; $3N_r$ computations for $\sum_{k=1}^{N_r} (\hat{u}_k^p)^3 B_{k,j}^{(2)}$, and $3(N_r^2 - N_r)$ operations for $\sum_{\substack{k,l=1 \\ k\neq l}}^{N_r} (\hat{u}_k^p)^2 \hat{u}_l^p C_{k,l,j}$, and $\frac{1}{6} N_r(N_r - 1)(N_r - 2)$ operations for $\sum_{\substack{i,k,l=1 \\ i<k<l}}^{N_r} \hat{u}_i^p \hat{u}_k^p \hat{u}_l^p D_{i,k,l,j}$. Hence, as required in the many-query or real-time contexts, the online complexity is independent of $\dim(V_{N,0}(\Omega))$, the dimension of the underlying spectral element approximation space. Since $N_r \ll \dim(V_{N,0}(\Omega))$, the cost to compute $\hat{u}_j^{p+1}$ in the online stage will typically be much less than the cost to directly evaluate $u_{N_r}^{p+1}$ in physical space.

Now the fine propagator $\mathcal{F}_{\Delta T}^{\rm se}$ is defined by (3.8), the coarse propagator $\mathcal{G}_{\Delta T}^{\rm serb}$ is defined by (4.3), and the iterative process (2.6) is of the form

$$\mathrm{u}_{n+1,\rm serb}^{k+1} = \mathcal{G}_{\Delta T}^{\rm serb}(\mathrm{u}_{n,\rm serb}^{k+1}) + \mathcal{F}_{\Delta T}^{\rm se}(\mathrm{u}_{n,\rm serb}^{k}) - \mathcal{G}_{\Delta T}^{\rm serb}(\mathrm{u}_{n,\rm serb}^{k}). \tag{4.5}$$

4.2. Semi-implicit finite element reduced basis approximation

The finite element reduced basis approximation $u_{N_s}^{q+1}(x, y)$ to $u(x, y, t^{q+1})$ is as the solution of

$$\begin{cases} (\partial_{\Delta t_{\rm fe}} u_{N_s}^q, v_{N_s})_h + (\partial_x u_{N_s}^q, v_{N_s})_h + \mu(\nabla A_{\Delta t_{\rm fe}} u_{N_s}^q, \nabla v_{N_s})_h \\ \qquad\qquad + (\phi(u_{N_s}^q), v_{N_s})_h = (f^q, v_{N_s})_h, \quad \forall v_{N_s} \in W_{N_s}^{\rm fe}, \forall q \in \mathcal{I}_{\rm fe}, \\ u_{N_s}^0 = \Pi_h^2 u_0. \end{cases} \tag{4.6}$$

Also, the finite element reduced basis functions are the orthogonalized with respect to the discrete inner product $(\cdot, \cdot)_h + \frac{1}{2}\mu\Delta t_{\rm fe}(\nabla\cdot, \nabla\cdot)_h$. Let $\{\lambda_i\}_{i=1,2,\cdots,N_s}$ be the orthogonal basis

functions of $W_{N_s}^{\rm fe}$ and $u_{N_s}^q = \sum_{k=1}^{N_s} \hat{u}_k^q \lambda_k$. From

$$(\lambda_k, \lambda_j)_N + \frac{1}{2}\mu\Delta t_{\rm se}(\nabla\lambda_k, \nabla\lambda_j)_N = \delta_{k,j},$$

we still have that

$$\begin{aligned}\hat{u}_j^{q+1} =& 2(u_{N_s}^q, \lambda_j)_h - \hat{u}_j^q - \Delta t_{\rm fe}(\partial_x u_{N_s}^q, \lambda_j)_h \\ & - \Delta t_{\rm fe}(\phi(u_{N_s}^q), \lambda_j)_h + \Delta t_{\rm fe}(f^q, \lambda_j)_h, \quad j = 1, \cdots, N_s. \end{aligned} \tag{4.7}$$

Off-line Stage

1. Calculate $\zeta_i^{\rm fe} = u_h^{n_i}$, $i = 1, \cdots, N_s$, to form $W_{N_s}^{\rm fe}$ as in (4.2) by using P_2-FEM.
2. Form the orthogonal basis functions $\{\lambda_i\}_{i=1,2,\cdots,N_s}$ of $W_{N_s}^{\rm fe}$.
3. Evaluate and store $B_{i,j}^{(0)} = 2(\lambda_i, \lambda_j)_h$, $B_{i,j}^{(1)} = \Delta t_{\rm fe}(\partial_x\lambda_i, \lambda_j)_h$, $i, j = 1, \cdots, N_s$.
4. Calculate and store $B_{i,j}^{(2)} = \Delta t_{\rm fe}(\lambda_i^3, \lambda_j)_h$, $C_{i,j,k} = 3\Delta t_{\rm fe}(\lambda_i^2\lambda_j, \lambda_k)_h$ and $D_{i,j,k,l} = 6\Delta t_{\rm fe}(\lambda_i\lambda_j\lambda_k, \lambda_l)_h$.
5. Compute and store the linear functions $F_j^{(q)} = \Delta t_{\rm fe}(f^q, \lambda_j)_h$, $j = 1, \cdots, N_s$, $\forall q \in P^{\rm fe}$.

On-line Stage

Denote $A_j^{(0,q)} = 2(u_{N_s}^q, \lambda_j)_h$, $A_j^{(1,q)} = \Delta t_{\rm fe}(\partial_x u_{N_s}^q, \lambda_j)_h$, and $A_j^{(2,q)} = \Delta t_{\rm fe}(\phi(u_{N_s}^q), \lambda_j)_h$.

1. Compute $A_j^{(0,q)} = \sum_{k=1}^{N_s} \hat{u}_k^q B_{k,j}^{(0)}$, $A_j^{(1,q)} = \sum_{k=1}^{N_s} \hat{u}_k^q B_{k,j}^{(1)}$, $j = 1, \cdots, N_s$.
2. Evaluate

$$A_j^{(2,q)} = \sum_{k=1}^{N_s} (\hat{u}_k^q)^3 B_{k,j}^{(2)} + \sum_{\substack{k,l=1 \\ k\neq l}}^{N_s} (\hat{u}_k^q)^2 \hat{u}_l^q C_{k,l,j} + \sum_{\substack{i,k,l=1 \\ i<k<l}}^{N_s} \hat{u}_i^q \hat{u}_k^q \hat{u}_l^q D_{i,k,l,j}, \ j = 1, \cdots, N_s.$$

3. Calculate $\hat{u}_j^{q+1} = A_j^{(0,q)} - \hat{u}_j^q - A_j^{(1,q)} - A_j^{(2,q)} + F_j^{(q)}$, $j = 1, \cdots, N_s$.

The operations in the online stage is thus only $\frac{1}{6}N_s^3 + \frac{5}{2}N_s^2 + \frac{7}{3}N_s$ for each j, and thus the online complexity is independent of $\dim(V_{h,0}^2(\Omega))$. Since $N_s \ll \dim(V_{h,0}^2(\Omega))$, we expect significant computational savings in the online stage relative to classical discretization and solution approaches.

Finally, the fine propagator $\mathcal{F}_{\Delta T}^{\rm fe}$ is defined by (3.11) with $s = 2$, the coarse propagator $\mathcal{G}_{\Delta T}^{\rm ferb}$ is defined by (4.6), and the iterative process (2.6) is of the form

$$\mathrm{u}_{n+1,\rm ferb}^{k+1} = \mathcal{G}_{\Delta T}^{\rm ferb}(\mathrm{u}_{n,\rm ferb}^{k+1}) + \mathcal{F}_{\Delta T}^{\rm fe}(\mathrm{u}_{n,\rm ferb}^{k}) - \mathcal{G}_{\Delta T}^{\rm ferb}(\mathrm{u}_{n,\rm ferb}^{k}). \tag{4.8}$$

5. Multi-grids and Multi-Degrees Finite Element Method

The coarse propagators $\mathcal{G}$ can be differ in other ways. In this section we consider other methods for the parareal in time algorithm of (3.2).

5.1. Semi-implicity multi-grids finite element approximation

We divide $\bar{I}$ into M_H subintervals of equal length $\Delta t_H = T/M_H$ and define $t^p = p\Delta t_H$, $0 \le p \le M_H \equiv T/\Delta t_H$, and $I_H \equiv \{1, 2, \cdots, M_H\}$. Also, we divide $\bar{I}$ into M_h subintervals of equal length $\delta t_h = T/M_h$ and define $t^q = q\delta t_h$, $0 \le q \le M_h \equiv T/\delta t_h$, and $I_h \equiv \{1, 2, \cdots, M_h\}$.

The coarse operator $\mathcal{G}^{\rm mg}_{\Delta T}$ based on larger time step Δt_H and the coarse grid (H) is defined as: find $u_H^{p+1} \in V^1_{H,0}(\Omega)$ such that

$$\begin{cases} (\partial_{\Delta t_H} u_H^p, v_H)_H + (\partial_x u_H^p, v_H)_H + \mu(\nabla A_{\Delta t_H} u_H^p, \nabla v_H)_H \\ \qquad\qquad +(\phi(u_H^p), v_H)_H = (f^p, v_H)_H, \quad \forall v_H \in V^1_{H,0}(\Omega), \forall p \in I_H, \\ u_H^0 = \Pi^1_H u_0. \end{cases} \tag{5.1}$$

Then we dive each coarse triangle into nine triangles to form a fine grid (h), in which the resulting number of vertices is equal to 2601 in the coarse mesh (H) and 22801 in the fine mesh (h). The propagator $\mathcal{F}^{\rm mg}_{\Delta T}$ based on time step δt and the fine grid (h) is defined as: find $u_h^{m+1} \in V^1_{h,0}(\Omega)$ such that

$$\begin{cases} (\partial^{pre}_{\delta t_h} u_h^m, v_h)_h + (\partial_x u_h^m, v_h)_h + \mu(\nabla A^{pre}_{\delta t_h} u_h^m, \nabla v_h)_h \\ \qquad\qquad +(\phi(u_h^m), v_h)_h = (f^m, v_h)_h, \qquad\qquad \forall v_h \in V^1_{h,0}(\Omega), \\ (\partial_{\delta t_h} u_h^m, v_h)_h + (\partial_x A^{pre}_{\delta t_h} u_h^m, v_h)_h + \mu(\nabla A_{\delta t_h} u_h^m, \nabla v_h)_h \\ \qquad\qquad +(A^{pre}_{\delta t_h}\phi(u_h^m), v_h)_h = (A_{\delta t_h} f^m, v_h)_h, \ \ \forall v_h \in V^1_{h,0}(\Omega), \forall m \in I_h, \\ u_h^0 = \Pi^1_h u_0. \end{cases} \tag{5.2}$$

Now the iterative process (2.6) is of the form

$$\mathrm{u}^{k+1}_{n+1,\rm mg} = \Pi^h_H \mathcal{G}^{\rm mg}_{\Delta T}(\Pi^H_h \mathrm{u}^{k+1}_{n,\rm mg}) + \mathcal{F}^{\rm mg}_{\Delta T}(\mathrm{u}^k_{n,\rm mg}) - \Pi^h_H \mathcal{G}^{\rm mg}_{\Delta T}(\Pi^H_h \mathrm{u}^k_{n,\rm mg}). \tag{5.3}$$

The operator that allows one to go from the coarse mesh to the fine one (denoted as Π^h_H) and reciprocally (i.e.,Π^H_h) is the interpolation operator.

5.2. Semi-implicity multi-degrees finite element approximation

We divide $\bar{I}$ into M_H subintervals of equal length $\Delta t_H = T/M_H$ and define

$$t^p = p\Delta t_H, 0 \le p \le M_H \equiv T/\Delta t_H, \text{and} I_H \equiv \{1, 2, \cdots, M_H\}.$$

Also, we divide $\bar{I}$ into M_{H^2} subintervals of equal length $\delta t_H = T/M_{H^2}$ and define $t^q = q\delta t_H$, $0 \le q \le q_H \equiv T/\delta t_H$, and $I_{H^2} \equiv \{1, 2, \cdots, M_{H^2}\}$.

The coarse operator $\mathcal{G}^{\rm md}_{\Delta T}$ based on larger time step Δt_H and the coarse grid (H) is defined as: find $u_H^{p+1} \in V^1_{H,0}(\Omega)$ such that

$$\begin{cases} (\partial_{\Delta t_H} u_H^p, v_H)_H + (\partial_x u_H^p, v_H)_H + \mu(\nabla A_{\Delta t_H} u_H^p, \nabla v_H)_H \\ \qquad\qquad +(\phi(u_H^p), v_H)_H = (f^p, v_H)_H, \quad \forall v_H \in V^1_{H,0}(\Omega), \ \ \forall p \in I_H, \\ u_H^0 = \Pi^1_H u_0. \end{cases} \tag{5.4}$$

Then the propagator $\mathcal{F}^{\rm md}_{\Delta T}$ based on time step δt_H is defined as: find $u_H^{m+1} \in V^2_{H,0}(\Omega)$ such that

$$\begin{cases} (\partial^{pre}_{\delta t_H} u_H^m, v_H)_H + (\partial_x u_H^m, v_H)_H + \mu(\nabla A^{pre}_{\delta t_H} u_H^m, \nabla v_H)_H \\ \qquad\qquad +(\phi(u_H^m), v_H)_H = (f^m, v_H)_H, \qquad\qquad \forall v_H \in V^2_{H,0}(\Omega), \\ (\partial_{\delta t_H} u_H^m, v_H)_H + (\partial_x A^{pre}_{\delta t_H} u_H^m, v_H)_H + \mu(\nabla A_{\delta t_H} u_H^m, \nabla v_H)_H \\ \qquad\qquad +(A^{pre}_{\delta t_H}\phi(u_H^m), v_H)_H = (A_{\delta t_H} f^m, v_H)_H, \quad \forall v_H \in V^2_{H,0}(\Omega), \ \ \forall m \in I_{H^2}, \\ u_H^0 = \Pi^2_H u_0. \end{cases} \tag{5.5}$$

Now the iterative process (2.6) is of the form

$$\mathrm{u}_{n+1,\mathrm{md}}^{k+1} = P_1^2\mathcal{G}_{\Delta T}^{\mathrm{md}}(P_2^1\mathrm{u}_{n,\mathrm{md}}^{k+1}) + \mathcal{F}_{\Delta T}^{\mathrm{md}}(\mathrm{u}_{n,\mathrm{md}}^{k}) - P_1^2\mathcal{G}_{\Delta T}^{\mathrm{md}}(P_2^1\mathrm{u}_{n,\mathrm{md}}^{k}), \tag{5.6}$$

where P_1^2 is the L^2 prolongation operator from $V_{H,0}^1(\Omega)$ onto $V_{H,0}^2(\Omega)$ and P_2^1 is the L^2 projection operator from $V_{H,0}^2(\Omega)$ onto $V_{H,0}^1(\Omega)$.

6. Numerical Results

In this section, we report the numerical results. Let $\mathrm{u}_{\cdot,N}$ be the solution of spectral element approximation (3.8), $\mathrm{u}_{\cdot,H}$ be the solution of finite element approximation (3.11) based on the grid (H) and $s = 2$, and $\mathrm{u}_{\cdot,h}$ be the solution of finite element method (5.2) based on the grid (h) and $s = 1$; and denoted by $\mathrm{u}_{\cdot,N_r}$ and $\mathrm{u}_{\cdot,N_s}$ the solution of scheme (4.3) based on the N_r dimensional spectral element reduced basis space and scheme (4.6) based on the N_s dimensional finite element reduced basis space, respectively. Denote by $\mathrm{u}_{n,\cdot}$ the numerical solution at time $t = T_n$. Define the relative error as

$$\mathrm{e}_{n,H,N_s}^{\mathrm{rb}} = \frac{\|\mathrm{u}_{n,H} - \mathrm{u}_{n,N_s}\|}{\|\mathrm{u}_{n,H}\|}, \qquad \mathrm{e}_{n,H,N_s}^{\mathrm{ferb},(k)} = \frac{\|\mathrm{u}_{n,H} - \mathrm{u}_{n,\mathrm{ferb}}^{(k)}\|}{\|\mathrm{u}_{n,H}\|},$$
$$\mathrm{e}_{n,H}^{\mathrm{md},(k)} = \frac{\|\mathrm{u}_{n,H} - \mathrm{u}_{n,\mathrm{md}}^{(k)}\|}{\|\mathrm{u}_{n,H}\|}, \qquad \mathrm{e}_{n,N,N_r}^{\mathrm{rb}} = \frac{\|\mathrm{u}_{n,N} - \mathrm{u}_{n,N_r}\|}{\|\mathrm{u}_{n,N}\|},$$
$$\mathrm{e}_{n,N,N_r}^{\mathrm{serb},(k)} = \frac{\|\mathrm{u}_{n,N} - \mathrm{u}_{n,\mathrm{serb}}^{(k)}\|}{\|\mathrm{u}_{n,N}\|}, \qquad \mathrm{e}_{n,h}^{\mathrm{mg,(k)}} = \frac{\|\mathrm{u}_{n,h} - \mathrm{u}_{n,\mathrm{mg}}^{(k)}\|}{\|\mathrm{u}_{n,h}\|}.$$

Example 6.1. Consider Eq.(3.1) on the unit square $\Omega = (-1, 1)^2$, with $\mu = 1.0$ and $T = 1$, choose the right-hand side $f = 0$ and the initial condition $u_0 = \sin(\pi x)\sin(\pi y)$. First, we test the convergence of the iterative process (5.3) and (5.6) based on the multi-grids and the multi-degrees FEM, respectively. We then compare the CPU time between different coarse solvers.

We consider the parareal in time algorithm based on the multi-grids and the multi-degrees FEM. For this example, a regular triangular mesh is used in computations. We choose the number of vertices is equal to 2601 in the coarse mesh (H) and 22801 in the fine mesh (h); take the coarse time step $\Delta t_H = 0.001$ in scheme (5.1) and (5.4), the fine time step $\delta t_h = \delta t_H = 0.001$ in scheme (5.2) and (5.5), and the time step of snapshots $\Delta T = 0.1$ in (5.3) and (5.6), respectively.

Numerical tests at time $T = 1$ are presented in Table 1. It indicates that both of the iterative process (5.3) and (5.6) provide a converging sequence. Indeed, we know from (2.7) that the errors between $\mathrm{u}_{n,\mathrm{mg}}^k$ and $u_h(T_n)$ (the solution of scheme (3.10) at time $t = T_n$ based on the fine mesh (h) and $s = 1$) is about $\varepsilon^k + \delta t_h^2$, while the errors between $\mathrm{u}_{n,\mathrm{md}}^k$ and $u_H(T_n)$ (the solution of scheme (3.10) at time $t = T_n$ based on the mesh (H) and $s = 2$) is about and $\varepsilon^k + \delta t_H^2$, respectively. Thus the numerical experiment supports the theoretical analysis.

We then compare the CPU time between the coarse propagator based on the finite element reduced basis scheme (4.6) and the coarse propagator based on the finite element scheme (5.1) with the mesh (H) or the mesh (h). We choose $\Delta t_{\mathrm{fe}} = \Delta t_H = 0.001$. The reduced basis approximation space is define as

$$W_{N_s}^{\mathrm{fe}} = \mathrm{span}\Big\{\zeta_1^{\mathrm{fe}} = \mathrm{u}_1, \zeta_2^{\mathrm{fe}} = \mathrm{u}_2, \cdots, \zeta_{N_s}^{\mathrm{fe}} = \mathrm{u}_{N_s}\Big\}, \tag{6.1}$$

Table 1. Errors of iterative process (5.3) and (5.6).

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$\mathrm{e}_{10,h}^{\mathrm{mg},(k)}$	3.7E-1	1.3E-2	5.8E-4	2.5E-4	1.9E-5
$\mathrm{e}_{10,H}^{\mathrm{md},(k)}$	4.5E-1	1.4E-2	2.0E-3	6.9E-6	2.7E-6

Table 2. CPU time of Scheme (4.6) and Scheme (5.1).

	Scheme(4.6)			Scheme(5.1)
	$N_s=3$	$N_s=4$	$N_s=5$	—
m.g.(H)	1.903(s)	2.614(s)	3.815(s)	147.242(s)
m.g.(h)	1.903(s)	2.614(s)	3.815(s)	580.175(s)

where u_n is the solution of the scheme (3.11) based on the mesh (H) or the mesh (h), $\delta t_{\mathrm{fe}} = 0.001$ and $s = 2$ at time $t = n\Delta T$.

Table 2 shows that one of the expensive part in the parareal in time algorithm is the resolution of the coarse solver computed in physical space, such as the SEM and the FEM, etc.. Also, it shows that the CPU time of our reduced basis approximation is dependent only on N_s (the dimension of reduced basis space), and independent of the mesh parameters, such as the vertices, the nodes and the dimension of the finite element approximation space. This is consistent with the analysis of the previous section. Thus, the reduced basis method provides a coarse and very cheap propagator.

Example 6.2. Consider Eq. (3.1) on the domain

$$\Omega = \left\{(x,y)|\ x^2+y^2 \geq \frac{1}{4},\ |x| \leq 1,\ |y| \leq 1\right\},$$

with $\mu = 1.0$ and $T = 10$, choose the right-hand side $f = 5\sin(3t+2)$ and the initial condition $u_0 = \sin(x^2+y^2-0.25)\sin(\pi x)\sin(\pi y)$. First, we test the convergence of the iterative process (4.5) for fixed N_r. We then show the accuracy of (4.5) decreases when N_r increases. Finally, we compare the accuracy of the iterative process (4.5) with the iterative process (4.8).

We now consider the spectral element approximation (3.8) and choose $\delta t_{\mathrm{se}} = 0.001$. We divide the computational domain Ω into E non-overlapping subdomains. Several spectral element meshes are presented in Fig. 6.1. Let $\Delta T = 0.1$. The reduced basis approximation space is define as

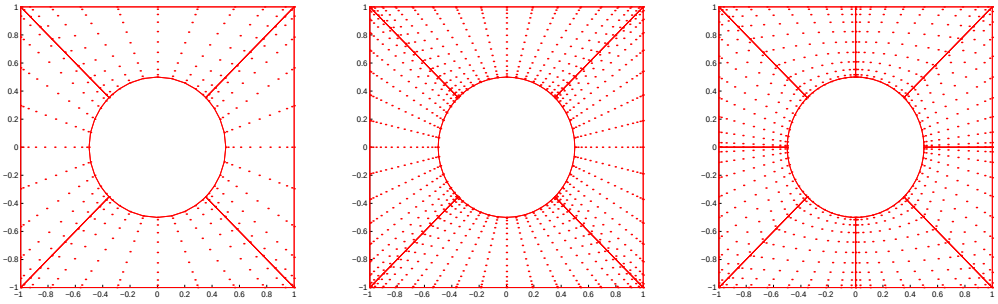
$$W_{N_r}^{\mathrm{se}} = \mathrm{span}\left\{\zeta_1^{\mathrm{se}} = \mathrm{u}_1, \zeta_2^{\mathrm{se}} = \mathrm{u}_2, \cdots, \zeta_{N_r}^{\mathrm{se}} = \mathrm{u}_{N_r}\right\}, \tag{6.2}$$

where u_n is the solution of the scheme (3.8) at time $t = n\Delta T$.

We choose $N_r = 3$ and $\Delta t_{\mathrm{se}} = 0.01$. The numerical tests are presented in Table 3. It indicates that the iterative process (4.5) is convergent, and the error $\|\mathrm{u}_{n,\mathrm{serb}}^k - \mathrm{u}_{n,N}\|$ is independent

Table 3. Errors of (4.5) for different E and N

(E,N)	$\mathrm{e}_{100,N,3}^{\mathrm{rb}}$	$\mathrm{e}_{100,N,3}^{\mathrm{serb},(1)}$	$\mathrm{e}_{100,N,3}^{\mathrm{serb},(2)}$	$\mathrm{e}_{100,N,3}^{\mathrm{serb},(3)}$	$\mathrm{e}_{100,N,3}^{\mathrm{serb},(4)}$	$\mathrm{e}_{100,N,3}^{\mathrm{serb},(5)}$
(4,16)	1.4E-3	5.4E-5	1.4E-6	5.5E-8	1.3E-8	1.4E-9
(4,10)	1.4E-3	5.4E-5	1.4E-6	5.6E-8	1.3E-8	1.4E-9
(8,10)	1.4E-3	5.4E-5	1.4E-6	5.5E-8	1.3E-8	1.4E-9



(a) (b) (c)

Fig. 6.1. Spectral element meshes with (a)$(E, N) = (4, 10)$, (b)$(E, N) = (4, 16)$, (c)$(E, N) = (8, 10)$.

Table 4 Errors of reduced basis method and iterative process (4.5)

	$N_r = 2$		$N_r = 3$		$N_r = 4$	
$\Delta t_{\rm se}$	0.01	0.001	0.01	0.001	0.01	0.001
$\mathrm{e}^{\rm rb}_{100,10,N_r}$	2.9E-2	2.9E-2	1.4E-3	1.4E-3	2.4E-4	3.9E-5
$\mathrm{e}^{\rm serb,(1)}_{100,10,N_r}$	4.8E-4	4.3E-4	5.4E-5	5.3E-5	6.0E-6	6.9E-7
$\mathrm{e}^{\rm serb,(2)}_{100,10,N_r}$	1.1E-5	1.1E-5	1.4E-6	1.4E-6	2.4E-7	1.6E-8
$\mathrm{e}^{\rm serb,(3)}_{100,10,N_r}$	3.5E-7	3.5E-7	5.6E-8	5.8E-8	7.4E-9	1.3E-9
$\mathrm{e}^{\rm serb,(4)}_{100,10,N_r}$	1.3E-8	1.3E-8	1.3E-8	1.3E-8	1.3E-10	1.7E-10
$\mathrm{e}^{\rm serb,(5)}_{100,10,N_r}$	2.2E-9	2.2E-9	1.4E-9	1.4E-9	1.1E-11	1.6E-11

of E and N. Moreover, Table 4 shows that the error $\|\mathrm{u}^k_{n,\rm serb} - \mathrm{u}_{n,N}\|$ is also independent of $\Delta t_{\rm se}$. Also, Table 4 indicates that when N_r increases, the solution of the reduced basis approximation converges at an exponential rate. Thus, the reduced basis method provides a technique how to speed up the iterative process (4.5). By compare with (2.7), we discover that the errors between $\mathrm{u}^k_{n,\rm serb}$ and $u_N(T_n)$ (the solution of scheme (3.7) at time $t = T_n$) is about

$$\begin{aligned}\|\mathrm{u}^k_{n,\rm serb} - u_N(T_n)\| &\le \|\mathrm{u}^k_{n,\rm serb} - \mathrm{u}_{n,N}\| + \|\mathrm{u}_{n,N} - u_N(T_n)\| \\ &\approx \varepsilon^k(N_r) + \delta t^2_{\rm se}, \end{aligned} \tag{6.3}$$

where $\varepsilon(N_r)$ decreases when N_r increases. This is consistent with the numerical results of $(N_r, k) = (2, 3), (3, 2), (4, 1)$.

We now choose $\delta t_{\rm se} = \delta t_{\rm fe} = 0.01$, $\Delta t_{\rm se} = \Delta t_{\rm fe} = 0.001$, $N_r = N_s = 3$, $(E, N) = (4, 10)$ and the mesh (h): (Triangle=6642, Vertices=3491, Nodes=13624). The reduced basis approximation space $W^{\rm fe}_{N_s}$ is defined in (6.1). Numerical results of scheme (4.3) and scheme (4.6), and iterative process (4.5) and iterative process (4.8) at time $t = n\Delta T$ are presented in Table 5. It indicates that iterative process (4.5) and iterative process (4.8) provide fast and very cheap iterative processes.

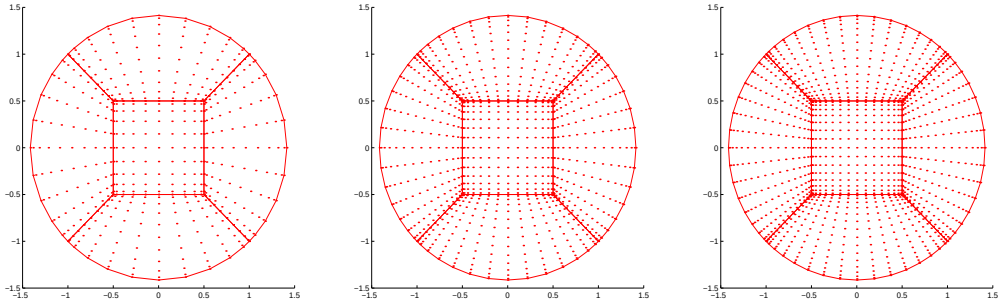
Example 6.3. Consider Eq. (3.1) on the domain

$$\Omega = \{(x, y)|\ x^2 + y^2 \le 2\},$$

with $\mu = 1.0$ and $T = 10$, choose the right-hand side $f = 3\cos(1.5t + 1) + 1$ and the initial condition $u_0 = \sin(x^2 + y^2 - 2)$. Again we compare the accuracy of the iterative process (4.5) with the iterative process (4.8).

Table 5. Errors of reduced basis methods and iterative processes (4.5) and (4.8)

n	$\mathrm{e}^{\mathrm{rb}}_{n,N,3}$	$\mathrm{e}^{\mathrm{serb},(1)}_{n,N,3}$	$\mathrm{e}^{\mathrm{serb},(3)}_{n,N,3}$	$\mathrm{e}^{\mathrm{serb},(5)}_{n,N,3}$	$\mathrm{e}^{\mathrm{rb}}_{n,H,3}$	$\mathrm{e}^{\mathrm{ferb},(1)}_{n,H,3}$	$\mathrm{e}^{\mathrm{ferb},(3)}_{n,H,3}$	$\mathrm{e}^{\mathrm{ferb},(5)}_{n,H,3}$
20	8.1E-4	5.2E-5	1.9E-7	4.1E-10	1.8E-2	4.0E-3	1.6E-4	3.2E-6
40	8.9E-4	5.2E-5	1.5E-7	1.4E-10	1.9E-2	3.9E-3	1.1E-4	3.7E-6
60	8.9E-4	5.3E-5	1.1E-7	1.7E-10	2.0E-2	3.7E-3	5.8E-5	9.9E-6
80	9.8E-4	5.3E-5	5.8E-8	6.2E-10	2.3E-2	3.4E-3	8.8E-5	2.1E-5
100	1.4E-3	5.4E-5	5.6E-8	1.4E-9	2.9E-2	2.6E-3	3.9E-4	5.0E-5



(a) (b) (c)

Fig. 6.2. Spectral element meshes with (a)$N = 10$, (b)$N = 14$, and (c)$N = 16$.

The spectral element meshes with $(E, N) = (5, 10)$, $(5, 14)$ and $(5, 16)$ are plotted in Fig. 6.2. The finite element mesh (H) with number of triangles, number of vertices, number of nodes (6970, 3586, 14141), (10910, 5581, 22071) and (13452, 6867, 27185) are considered. The reduced basis approximation spaces $W^{\mathrm{se}}_{N_r}$ and $W^{\mathrm{fe}}_{N_s}$ are defined in (6.1) and (6.2), respectively. We choose $\Delta T = 0.1$, $\delta t_{\mathrm{se}} = \delta t_{\mathrm{fe}} = 0.001$, $\Delta t_{\mathrm{se}} = \Delta t_{\mathrm{fe}} = 0.01$ and $N_r = N_s = 4$. The Numerical testes are presented in Table 6. It indicates that iterative process (4.5) and iterative process (4.8) provide fast and very cheap iterative processes. It also shows that the error $\|\mathrm{u}^k_{n,\mathrm{serb}} - \mathrm{u}_{n,N}\|$ is independent of N, and the error $\|\mathrm{u}^k_{n,\mathrm{ferb}} - \mathrm{u}_{n,H}\|$ is independent of (H) when the mesh (H) fine enough.

Table 6. Errors of iterative processes (4.5) and (4.8)

N	10	14	16	Vertices	3586	5581	6867
$\mathrm{e}^{\mathrm{rb}}_{100,N,4}$	1.0E-2	1.0E-2	1.0E-2	$\mathrm{e}^{\mathrm{rb}}_{100,H,4}$	9.0E-3	9.0E-3	9.0E-3
$\mathrm{e}^{\mathrm{serb},(1)}_{100,N,4}$	1.2E-3	1.2E-3	1.2E-3	$\mathrm{e}^{\mathrm{ferb},(1)}_{100,H,4}$	9.2E-4	9.1E-4	9.2E-4
$\mathrm{e}^{\mathrm{serb},(2)}_{100,N,4}$	1.0E-4	1.0E-4	1.0E-4	$\mathrm{e}^{\mathrm{ferb},(2)}_{100,H,4}$	4.1E-4	2.7E-4	2.2E-4
$\mathrm{e}^{\mathrm{serb},(3)}_{100,N,4}$	1.0E-5	1.0E-5	1.0E-5	$\mathrm{e}^{\mathrm{ferb},(3)}_{100,H,4}$	2.9E-4	1.8E-4	1.5E-4
$\mathrm{e}^{\mathrm{serb},(4)}_{100,N,4}$	1.6E-6	1.6E-6	1.6E-6	$\mathrm{e}^{\mathrm{ferb},(4)}_{100,H,4}$	1.8E-4	1.2E-4	9.7E-5
$\mathrm{e}^{\mathrm{serb},(5)}_{100,N,4}$	3.4E-7	3.4E-7	3.4E-7	$\mathrm{e}^{\mathrm{ferb},(5)}_{100,H,4}$	1.1E-4	7.2E-5	5.8E-5

7. Conclusions

In this paper, we have presented the reduced basis technique as a coarse solver for the parareal in time algorithms of nonlinear evolutionary parabolic partial differential equations. We have demonstrated that a coarse solver based on reduced basis technique provides higher

accuracy and significant speedup than the multi-grids finite element method and the multi-degrees finite element method during the parareal in time simulation procedure. The methods presented in this paper have good efficiency for computing the solutions of the ordinary and partial differential equations and also provide a way to construct parallel method.

Although we have only considered the nonlinear evolutionary parabolic partial differential equations, the proposed methods can also be efficiently applied to the Navier-Stokes equations. On the other hand, the proposed method has also its limitations. The method is efficient when applied to problems approximated by a continuous Galerkin method; however, for the discontinuous Galerkin method (see [7,27]), there are still some challenging issues.

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