The Traveling Wave of Auto-Catalytic Systems-Monotone and Multi-Peak Solutions

Yuanwei Qi *

Department of Mathematics, University of Central Florida, Orlando, Florida 32816, USA

Received 31 March, 2016; Accepted 15 May, 2016

Abstract. This article studies propagating wave fronts of a reaction-diffusion system modeling an isothermal chemical reaction \( A + 2B \to 3B \) involving two chemical species, a reactant \( A \) and an auto-catalyst \( B \), whose diffusion coefficients, \( D_A \) and \( D_B \), are unequal due to different molecular weights and/or sizes. Explicit bounds \( c_* \) and \( c^* \) that depend on \( D_B / D_A \) are derived such that there is a unique travelling wave of every speed \( c \geq c_* \) and there does not exist any travelling wave of speed \( c < c_* \). Furthermore, the reaction-diffusion system of the Gray-Scott model of \( A + 2B \to 3B \), and a linear decay \( B \to C \), where \( C \) is an inert product is also studied. The existence of multiple traveling waves which have distinctive number of local maxima or peaks is shown. It shows a new and very distinctive feature of Gray-Scott type of models in generating rich and structurally different traveling pulses.

AMS subject classifications: 34C20, 34C25, 92E20

Key words: Qubic autocatalysis, travelling wave, minimum speed, Gray-Scott, multi-peak waves.

1 Introduction

Autocatalytic chemical reaction of the form

\[ A + nb \to (n+1)b \text{ with rate } kab^n \text{ and } n = 1, 2 \]

between two chemical species \( A \) and \( B \), appears in many chemical wave models of excitable media from the idealized Brusselator to real-world clock reactions such as Belousov-Zhabotinsky reaction, the Briggs-Rauscher reaction, the Bray-Liehafsky reaction and the iodine clock reaction. In that setting, their importance was recognized pretty early [13, 14, 29].

*Corresponding author. Email address: Yuanwei.Qi@ucf.edu (Y. Qi)
More recently, in various models of biological pattern formation of Turing type, for the purpose of replicating experimental results in early 1990s, whether it is CIMA or Gray-Scott [20, 22] chemical reaction of the form

$$ A + 2B \rightarrow 3B \quad \text{and} \quad B \rightarrow C, $$

with C an inert chemical species, plays a significant role. In particular, in Gray-Scott model with feeding, self-replicating traveling pulse (traveling wave) is the most exciting and not completely understood phenomenon [9–11, 18].

In this work, we study the traveling wave problem of autocatalytic chemical reaction

$$ A + nB \rightarrow (n+1)B, $$

which, after simple non-dimensionalization results in the reaction-diffusion system,

$$ \begin{cases} 
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^n, \\
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + uv^n,
\end{cases} \tag{I} $$

as well as that of chemical reaction

$$ A + nB \rightarrow (n+1)B, \quad \text{and} \quad B \rightarrow C, $$

which has the governing equations

$$ \begin{cases} 
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^n, \\
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + uv^n - kv^m.
\end{cases} \tag{II} $$

Here, $D$ a positive constant is the ratio of diffusion coefficients of chemical species $B$ to that of $A$, $n \geq 1$ is a positive constant not necessarily an integer, and $kv^m$ describes the rate of $B \rightarrow C$, with $k$ and $m \geq 1$ both positive constants. We assume throughout that $1 \leq m \leq n$.

For a traveling wave solution to (I), $u(x,t) = u(z), \ v(x,t) = v(z)$, where $z = x - ct$, the governing ODE system is:

$$ \begin{cases} 
u'' + cu' - uv^n = 0, \\
Dv'' + cv' + uv^n = 0, \tag{1.1}
\end{cases} $$

where $c > 0$ is a constant. Assuming

$$ \lim_{z \rightarrow -\infty} (u,v) = (0,a), \quad a > 0, $$

the addition of the two equations and integration on $(-\infty, z]$ yield

$$ u' + Dv' + c(u + v - a) = 0. \tag{1.2} $$
This, with the easy to verify property that \( u \) is increasing before \( v \) hits its first zero, implies that for a traveling wave solution,
\[
\lim_{z \to \infty} (u, v) = (a, 0).
\]

By a simple scaling, we only need to consider \( a = 1 \), and use (1.2) to replace the first equation in (1.1). The traveling wave problem of (I) is the following:
\[
\begin{align*}
\frac{d}{dz} u' + Dv' &= c(1-u-v), \quad u' \geq 0 \quad \forall z \in \mathbb{R}, \\
Dv'' + cv' &= -uv^n, \quad v \geq 0 \quad \forall z \in \mathbb{R}, \\
\lim_{z \to \infty} (u, v) &= (1, 0), \\
\lim_{z \to -\infty} (u, v) &= (0, 1).
\end{align*}
\]

(1.3)

It turns out the traveling wave problem of (I) is of the mono-stable type for scalar equation with a minimum positive speed. The main effort is then to prove sharp bound on minimum speed, because the minimum speed traveling wave is the most stable one. The main results on (1.3) are summarized in the following two theorems, which show \( D > 1 \) is very different from \( D < 1 \).

**Theorem 1.1.** Suppose \( D < 1 \) and \( n > 1 \). There exists no traveling wave of (1.3) if
\[
c \leq \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{D}{1}) \frac{\sqrt{4K(n)+1-1}}{\sqrt{4K(n)+1+1}}}}
\]
where \( K(n) \) is a constant which depends on \( n \) only and is an increasing function of \( n \). In particular, \( K(1) = 1/4, K(2) = 2 \). For existence, we have the following results.

(i) If \( n \geq 2 \), a unique (up to translation) traveling wave solution exists for (I) for each
\[
c \geq 4D/\sqrt{1+4D},
\]
(ii) If \( 1 < n < 2 \), a unique (up to translation) traveling wave solution exists for (I) for each
\[
c \geq \frac{2D}{(-D^2+n^2)^{1/2}},
\]
where
\[
v = \frac{n-1+\sqrt{(n-1)^2+8(3-n)D+16D^2}}{4}.
\]

**Theorem 1.2.** Suppose \( D \geq 1 \) and \( n \geq 1 \). There exists a positive constant \( c_{\text{min}} \) such that (I) has a traveling wave if and only if \( c \geq c_{\text{min}} \). In addition, \( c_{\text{min}} \) is bounded by
\[
\sqrt{\frac{D}{K(n)}} \leq c_{\text{min}} \leq \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{D}{1}) \frac{\sqrt{4K(n)+1-1}}{\sqrt{4K(n)+1+1}}}}
\]
where \( K(n) \) is the same constant as in Theorem 1.1.
Remark 1.1. It should be clear from the above two results that $c_{\text{min}}$ is of order $O(D)$ if $0 < D \ll 1$, but $O(\sqrt{D})$ if $D \gg 1$.

For system (II), since any equilibrium point is of the form $(a,0)$, $a \in \mathbb{R}$, the traveling wave problem takes the form

$$
\begin{cases}
  u'' + cu' = uv^n, & u' > 0 \quad \text{in } \mathbb{R}, \\
  Dv'' + cv' = kv^m - uv^n, & v > 0 \quad \text{in } \mathbb{R}, \\
  u(-\infty) = u_0, & v(-\infty) = 0, \quad v(\infty) = 0, \quad u(\infty) < \infty.
\end{cases}
$$

(1.4)

It is easy to show that the traveling wave solution to (II) exists only when $n \geq m$. The traveling wave problem of (II), when $n > m$, is very different from any of the main types of scalar equation as well as other related models such as (I) or the case of $n = m$, see [8] for more details. For simplicity, we shall only treat the case of $m = 1$ and $k = 1$. The case of $n > m > 1$ is dealt with in [32].

Theorem 1.3. Let $D > 0$, $n > 1$ and $k = m = 1$, and $u_0 > 0$ be given constants. There exists a positive constant $c$ such that (1.4) admits a solution. In addition, the set of speeds for existence lies in a bounded interval for a given value of $u_0 > 0$. Furthermore, the speed $c$ must satisfy

$$
c^2 < 2D \left[ \max(1,D) \left( \frac{n+1}{2u_0} \right)^{n/(n-1)} \frac{n+1}{n-1} + n - 1 \right].
$$

The most surprising result is the one which shows when $u_0 \gg 1$, we have a large number of traveling wave solutions, each with fixed number of local maxima for $w = uv^{n-1}$. For this, we re-write (1.4), after proper scaling, as

$$
\begin{cases}
  du'' + cu' = uv^n, & u' > 0 \quad \text{in } \mathbb{R}, \\
  dv'' + cv' = v - uv^n, & v > 0 \quad \text{in } \mathbb{R}, \\
  u(-\infty) = h, & v(-\infty) = 0, \quad v(\infty) = 0, \quad u(\infty) < \infty.
\end{cases}
$$

(1.5)

where $d = D^{-1}$. We make the following change of scale and variables:

$$
\varepsilon = h^{-\frac{n}{n-1}}, \quad u = [1 + \varepsilon U]h, \quad v = h^{\frac{1}{n-1}} V, \quad c = \varepsilon C.
$$

Then (1.4) is equivalent to finding $(U,V,C) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times (0,\infty)$ which satisfy

$$
\begin{cases}
  dU'' + C\varepsilon U' = [1 + \varepsilon U]\varepsilon^n, & U' > 0 \quad \text{in } \mathbb{R}, \\
  V'' + \varepsilon C V' = V - [1 + \varepsilon U]\varepsilon^n, & V > 0 \quad \text{in } \mathbb{R}, \\
  U(-\infty) = 0, & V(-\infty) = 0, \quad V(\infty) = 0, \quad U(\infty) < \infty.
\end{cases}
$$

(1.6)

Theorem 1.4. Let $n > 1$ and $D > 0$ be given constants.
1. There exist positive constants $M_1$, $M_2$, and $M_3$ that depend only on $n$ and $D$ such that for each $\varepsilon > 0$, (1.6) admits no solution if $C \geq \max\{\sqrt{M_1/\varepsilon}, M_2\}$ or if $C \leq \gamma - M_3\varepsilon$.

2. For each sufficiently small positive $\varepsilon$ and each integer $L$ satisfying $1 \leq L \leq \varepsilon^{-1/4}$, there exists a constant $C_L = L\gamma [1 + O(\varepsilon + [L-1]^2\varepsilon|\ln \varepsilon|)]$ such that when $C = C_L$, the system (1.6) admits a solution, unique up to a translation. The solution is an $L$-hump solution in the sense that $w = [1 + \varepsilon U^n] V^n - 1$ admits exactly $L$ local maxima and $L-1$ interior local minima. In addition, if denoting the interior points of local minima of $w$ by $\{a_i\}_{i=2}^L$ and points of local maxima by $\{b_i\}_{i=1}^L$ with $-\infty = a_1 < b_1 < a_2 < b_2 < \cdots < b_L < a_{L+1} = \infty$, then

$$w(b_i) = M + O(i[L + 1 - i]\varepsilon), \quad G(w(a_{i+1})) = i(L - i) \sigma \gamma \varepsilon + O(i^2 L^2 \varepsilon^2 |\ln \varepsilon|) \quad \forall i = 1, \cdots, L.$$  

Furthermore, $\|w^2 - G(w)\|_{L^\infty(\mathbb{R})} = O(L^2 \varepsilon)$ and

$$\lim_{\varepsilon \rightarrow 0} \frac{w(b_i + z)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{v(b_i + z)}{\varepsilon} = W(z)$$

uniformly in $i = 1, \cdots, L$ and locally uniformly in $z \in \mathbb{R}$, where $W$ is the unique solution of

$$W'' = W - W^n \quad \text{in} \quad \mathbb{R}, \quad W(0) = M, \quad W'(0) = 0, \quad (1.7)$$

where

$$G(s) = s^2 - \frac{2s^{n+1}}{n+1}, \quad \alpha = \frac{1}{n-1}, \quad M = \left(\frac{n+1}{2}\right)^\alpha, \quad \sigma = 4 \int_0^M \sqrt{G(s)} ds, \quad \gamma = \frac{2\alpha}{D} \int_0^M \frac{s^n ds}{\sqrt{G(s)}}. \quad (1.8)$$

$s_+ = \max\{s,0\}$.

For related works on existence, stability and global dynamics of (I) or (II), we refer the reader to [3, 12, 19, 21, 25, 28, 30, 31].

The organization of the paper is as follows. In section 2, we study the system (1.3), and in section 3 we analyze the systems (1.4) and (1.6).

2 On the system (1.3)

In this section we study (1.3) and prove Theorems 1.1 and 1.2. We note in passing that for $n = 1$, the minimum speed can be determined explicitly by linearized systems around the two equilibrium points $(0,1)$ and $(1,0)$. Heretofore, we only consider $n > 1$.

2.1 Basic properties of traveling waves

First, we list some easy to verify properties as follows.

**Proposition 2.1.** Any solution $(u,v)$ of (1.3) has the following properties:
1. $u' > 0 > v'$ on $\mathbb{R}$;

2. $u + v < 1$ on $\mathbb{R}$ if $D < 1$, $u + v = 1$ if $D = 1$, and $u + v > 1$ if $D > 1$;

3. in the phase-space of $(u,v,v')$, the equilibrium point $(0,1,0)$ is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold, while $(1,0,0)$ has two-dimensional stable manifold and a one-dimensional center manifold when $n > 1$.

Different from earlier works in [2, 23], here we shall use a transformation to turn (1.3), which is equivalent to a third order autonomous dynamical system, into a second order non-autonomous system, using $w := 1 - v$ as the independent variable. This is legitimate since, $v' < 0$, so $z \to 1 - v(z)$ has an inverse. To further simplify the system, we let

$$w = 1 - v, \quad Q = \frac{Du}{c^2}, \quad y = \frac{cz}{D}, \quad \sigma := \frac{D}{c}.$$  

The system (1.3), with $P(w) = w_y$, becomes

$$P' = \sigma^2[P + w] - DQ \quad \forall \ w \in [0,1],$$

$$(2.1)$$

Lemma 2.1. For every $D > 0$ and $\sigma > 0$, (2.1) has a unique solution. In addition,

$$P(w) = \lambda w + O(w^2), \quad Q(w) = \lambda(1 + \lambda)w + O(w^2) \quad \text{as} \ w \searrow 0$$  

(2.2)

where

$$\lambda := \frac{1}{2}\left(\sqrt{4\sigma^2 + D^2} - D\right) \quad \text{the only positive root to} \ \lambda(\lambda + D) = \sigma^2.$$  

Furthermore, $Q'(w) > 0$ for all $w \in [0,1)$ and there are only two possible cases:

(a) $P(1) > 0$; there does not exist any travelling wave solution to (1.3).

(b) $P(1) = 0$; there exists a travelling wave solution to (1.3), unique up to translation.

The detailed proof is carried out in [5], we omit it here.

Our approach is based on comparison with the scaler case of $D = 1$, for which the classical theory is readily available.

Next, we review the existence of traveling wave of unit speed to the equation

$$w_{zz} + w_z = kw(1 - w)^n, \quad 0 \leq w \leq 1 \quad \text{on} \ \mathbb{R}, \quad w(-\infty) = 0, \ w(\infty) = 1.$$  

(2.3)

Here $n \geq 1$ is a parameter and $k$ is a positive constant. We seek upper bounds on $k$ for the existence of a solution. Since a solution, if it exists, satisfies $w_z > 0$ on $\mathbb{R}$, we can write
It follows that there exists a traveling wave solution to
\[ w' = q(w) \] and work on the \((w,q)\) phase plane. The resulting equation on the phase plane is
\[
\begin{align*}
q' + q &= kw(1-w)^n \quad \forall \ w \in [0,1], \\
q(0) &= 0, \quad q > 0 \text{ on } (0,1).
\end{align*}
\tag{2.4}
\]

There is a one-to-one correspondence between solutions to (2.3) and solutions to (2.4) satisfying the additional requirement \(q(1) = 0\).

**Lemma 2.2.** For each \(n \geq 1\) and \(k > 0\), there exists a unique solution \(q = q(\cdot; n, k)\) to (2.4). In addition, there exists a positive constant \(K(n)\) such that \(q(1; n, k) = 0\) if \(k \in (0, K(n)]\) and \(q(1; n, k) > 0\) if \(k \in (K(n), \infty)\). Consequently, (2.4) admits a solution if and only if \(k \in (0, K(n)]\). Moreover, \(K(n)\) is a strictly increasing function of \(n\) and \(K(1) = \frac{1}{4}, \quad K(2) = 2\).

The proof is based on a standard comparison argument and can be found in [5].

### 2.2 The case \(D \geq 1\)

**Lemma 2.3.** Suppose \(D \geq 1\). Then \(DQ(w) \geq \sigma^2 w\) for all \(w \in [0,1]\). Consequently, there is no travelling wave solution to (1.3) when \(\sigma^2 > DK(n)\), i.e., when \(c < \sqrt{D/K(n)}\).

**Proof.** If \(D = 1\), \(Q(w) = \sigma^2 w\) for all \(w \in [0,1]\). When \(D > 1\), for every \(w \in (0,1)\),
\[
P[DQ - \sigma^2 w]' = -D[DQ - \sigma^2 w] + (D - 1)\sigma^2 P > -D[DQ - \sigma^2 w].
\]

Moreover, when \(0 < w < 1\), \(DQ(w) = D(1 + \lambda)w + O(w^2) > |D + \lambda|\lambda w = \sigma^2 w\). Applying the Gronwall’s inequality, we derive that \(DQ(w) > \sigma^2 w\) on \((0,1)\).

Now suppose \(\sigma^2 > DK(n)\). Let \(k \in (K(n), \sigma^2 / D)\). Then \(Q(w) \geq kw\) on \([0,1]\) so that
\[
PP' + P = Q(1-w)^n \geq kw(1-w)^n \quad \forall w \in [0,1].
\]

We compare \(P(w)\) and the solution \(q(w; n, k)\) given in Lemma 2.2. Using Taylor expansion we can show that \(P(w) > q(w; n, k)\) for all \(0 < w < \epsilon\) for some \(\epsilon > 0\). In the interval \([\epsilon, 1]\) we can use the regular comparison principle to show that \(P(w) > q(w; n, k)\) for all \(w \in [\epsilon, 1]\). In particular, \(P(1) \geq q(1; n, k) > 0\), so that there is no travelling wave solution to (1.3). Since \(\sigma = D/c\), the condition \(\sigma^2 > DK(n)\) is the same as \(c < \sqrt{D/K(n)}\). \(\square\)

**Lemma 2.4.** Suppose \(D > 1\). Then,
\[
Q(w) < \lambda(1+\lambda)w, \quad P(w) < \lambda w \quad \forall w \in (0,1).
\]

It follows that there exists a traveling wave solution to (1.3) if \(\lambda(\lambda + 1) \leq K(n)\), i.e., when
\[
c \geq \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{D}) \frac{\sqrt{4K(n)+1} - 1}{\sqrt{4K(n)+1}}}}.
\]
Proof. A higher order Taylor expansion near \( w = 0 \) shows that \( Q < \lambda (\lambda + 1) w \) and \( P < \lambda w \) for all sufficiently small positive \( w \). Set

\[
\hat{B} = \sup \{ b \in (0,1) \mid P(w) < \lambda w, \quad Q(w) < \lambda (1 + \lambda) w \quad \forall w \in (0,b) \}.
\]

We show that \( \hat{B} = 1 \). Suppose to the contrary that \( \hat{B} < 1 \). Then either \( P(\hat{B}) - \lambda \hat{B} = 0 \) or \( Q(\hat{B}) - \lambda (1 + \lambda) \hat{B} = 0 \). In \((0,\hat{B})\),

\[
P[Q - \lambda (1 + \lambda) w]' = \sigma^2 (P + w) - DQ - \lambda (1 + \lambda) P
\]

\[
= \lambda (D + \lambda) (P + w) - DQ - \lambda (1 + \lambda) P
\]

\[
= -D[Q - \lambda (1 + \lambda) w] + \lambda (D - 1)(P - \lambda w)
\]

\[
\leq -D[Q - \lambda (1 + \lambda) w].
\]

Gronwall’s inequality then implies that \( Q < \lambda (\lambda + 1) w \) on \((0,\hat{B})\). Similarly, for all \( w \in (0,\hat{B}) \),

\[
P[Q - \lambda (1 + \lambda) w]' = -(1 + \lambda) P + Q(1-w)^n
\]

\[
= -(1 + \lambda) (P - \lambda w) - \lambda (1 + \lambda) w + Q(1-w)^n
\]

\[
< -(1 + \lambda) (P - \lambda w).
\]

The Gronwall’s inequality shows that \( P < \lambda w \) on \((0,\hat{B})\). We reach a contradiction. This proves that \( \hat{B} = 1 \); i.e. \( P(w) < \lambda w \) and \( Q(w) < \lambda (1 + \lambda) w \) for all \( w \in (0,1) \).

Suppose \( \lambda (1 + \lambda) \leq K(n) \). We can use comparison to show that \( P(w) \leq q(w; n, K(n)) \) for all \( w \in [0,1] \) so that \( P(1) = 0 \). Namely, there exists a travelling wave solution to (1.3).

Proof of Theorem 1.1. The estimate of \( c_{\text{min}} \), when it exists, follows from the above two lemmas. The existence of \( c_{\text{min}} \) can be derived using a comparison argument, which is quite standard, see [5].

2.3 The case of \( D < 1 \)

Lemma 2.5. Suppose \( D < 1 \). Then

(i) \( Q(w)(1-w)^{n/2} \leq \lambda [P+w] \quad \forall w \in [0,1] \).

(ii) \( Q \geq \lambda (P+w) \) and \( \lambda w(1-w)^n \leq P \leq \lambda w \) in \([0,1] \).

The proof follows from considering the difference \( (1-w)^{n/2} Q - \lambda (P+w) \), for (i) and \( Q - \lambda (P+w) \) for (ii), we omit the details.

Lemma 2.6. Suppose \( D < 1 \). Then \( Q > \lambda (1+\lambda) w \) on \((0,1) \). Consequently, when \( \lambda (1+\lambda) > K(n) \), i.e.

\[
c < \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{n}) \sqrt{\frac{4K(n)+1-1}{\sqrt{4K(n)+1+1}}}}}
\]

there is no travelling wave solution to (1.3).
**Proof.** The lemma is proved by considering $Q - \lambda (1 + \lambda)w$ and the use of Lemma 2.5.

For the existence part of Theorem 1.2, Lemma 2.5 is sufficient for $n \geq 2$. But, for $1 < n < 2$, we need to derive additional estimate.

**Lemma 2.7.** Suppose $D < 1$ and $1 < n < 2$. Then, $Q(1-w)^{n-1} \leq \mu(P+w)$ in $[0,1]$ if

\[ \mu \geq \frac{\sigma^2}{\lambda(n-1)+D} > \lambda. \]

For detailed demonstration, see [6].

**Proof of Theorem 1.2.** The non-existence follows directly from Lemmas 2.6. We now prove the existence. Since the proof of $n \geq 2$ is similar to that of $1 < n < 2$, we only prove the latter case. Simple computation shows that

\[ c \geq \frac{2D}{(-D+v^2)^{1/2}} \iff \mu \equiv \sigma^2/[(n-1)\lambda+D] \leq 1/4. \]

We proceed to show that there exists $\eta > \mu$ such that $P - \eta w(1-w) \leq 0$ on $(0,1)$. It is easy to verify, using result of Lemma 2.7, that

\[ [P - \eta w(1-w)]' = -1 + \frac{Q}{P} (1-w)^n - \eta(1-2w) \]

\[ \leq -1 + \mu(1-w) + \frac{\mu w(1-w)}{P} - \eta(1-2w). \]

At the point where $P = \eta w(1-w)$, the right hand side equals to

\[ -1 + \mu(1-w) + \frac{\mu}{\eta} - \eta(1-2w). \]

Elementary computation shows that if $\mu \leq 1/4$, there exists an $\eta > \mu$ which makes the quantity in (2.5) negative for all $w \in (0,1)$. Hence, there exists $\eta > \mu$ such that $P - \eta w(1-w) \leq 0$ on $(0,1)$. In consequence, $P(1) = 0$. This proves the existence, and completes the proof of the theorem.

**3 On the system (1.4)**

Unlike the system (1.3), which can be reduced to a second order system, the system (1.4) has no conserved quantity and there is no explicit connection between the equilibrium point $(u_0,0)$ at $-\infty$ and the one at $\infty$. Moreover, $v$ loses its monotonicity. For $n = m$, it was demonstrated in [15,16,26,27] that any traveling wave has the property that $v$ is bell-shaped. But, when $n > m$, our results show that $v$ can oscillate a large number of times,
which increases the difficulty of rigorous analysis. In particular, when \( n = 2 \) and \( m = 1 \), it is the famous Gray-Scott model of pattern formation. Our results are a clear indicator why it has the ability to generate rich pattern and dynamics.

Due to limited space, we shall only give detailed proof of Theorem 1.3. The proof of Theorem 1.4 can be found in [4]. For convenience, we re-write (1.4) as

\[
\begin{align*}
    u'' + cu' &= uv^+_{n}, & u' > 0 & \quad \text{in } \mathbb{R}, \\
    Dv'' + cv' &= v - uv^+_{n}, & v > 0 & \quad \text{in } \mathbb{R}, \\
    u(-\infty) &= u_0, & v(-\infty) &= 0, & v(\infty) &= 0, & u(\infty) &< \infty,
\end{align*}
\]

(3.1)

where \( v^+ := \max\{v, 0\} \).

### 3.1 Preliminary

**Lemma 3.1.** For each \( c \geq 0 \), problem (3.1) admits a unique solution. The solution depends on \( c \) continuously and satisfies \( u' > 0 \) in \( \mathbb{R} \) and \( \lambda v - v' > 0 \), where \( \lambda = 2 / (\sqrt{c^2 + 4D} + c) \). In addition, define

\[
    x_1(c) := \sup\{z \in \mathbb{R} | v' > 0 \text{ in } (-\infty, z)\}, \quad x_2(c) := \sup\{z \in \mathbb{R} | v > 0 \text{ in } (-\infty, z)\};
\]

(3.2)

then \( x_1(c) < \infty, v'(x_1(c), c) = 0 > v''(x_1(c), c), \) and one and only one of the following holds:

1. \( x_2(c) < \infty \) and \( v(x_2(c), c) = 0 \);
2. \( x_2(c) = \infty \) and \( \lim_{x \to \infty} u(x, c) = \infty \);
3. \( x_2(c) = \infty \) and \( \lim_{x \to \infty} u(x, c) < \infty \). In this case, \( \lim_{x \to \infty} v(x, c) = 0 \), so \( (c, u, v) \) solves (1.4).

The proof is elementary and can be found in [8].

We define

\[
\begin{align*}
    A &:= \{c \geq 0 | x_2(c) < \infty\}, \\
    B &:= \{c \geq 0 | x_2(c) = \infty, \lim_{x \to \infty} u(x, c) = \infty\}, \\
    C &:= \{c \geq 0 | x_2(c) = \infty, \lim_{x \to \infty} u(x, c) < \infty\}. \tag{3.3}
\end{align*}
\]

We shall show that \( A \) and \( B \) are open, \( 0 \in A \), and \( [C, \infty) \subset B \) for some \( C \gg 1 \). Thus, \( C \) is non-empty and problem (1.4) admits a solution for some \( c > 0 \).

**Lemma 3.2.** The set \( A \) defined in (3.3) is open. In addition, \( 0 \in A \).
Proof. \(A\) is open is simple and standard. We show that \(0 \in A\). Assume that \(c = 0\). Recall that \(v'(x_1(0), 0) = 0 > v''(x_1(0), 0)\). We can define

\[
\hat{x}_2 = \sup\{z > x_1(0) \mid v'(\cdot, 0) < 0 \text{ in } (x_1(0), z)\}.
\]

Set \(u = u(x_1(0), 0)\). Since \(u' > 0\) in \(\mathbb{R}\), we find that \((u - u')v' \geq 0\) in \((-\infty, \hat{x}_2)\). Hence, integrating \(v'[Dv'' - v + \tilde{u}v^n] = v'[\tilde{u} - u]v^n\) we obtain, for each \(x < \hat{x}_2\),

\[
\frac{Dv'^2}{2} - \frac{v'^2}{2} + \frac{\tilde{u}v^{n+1}}{n+1} = \delta(x) := \int_{-\infty}^{x} [\tilde{u} - u(y)]v^n(y)v'(y)dy > 0.
\]

Note that \(\delta(x)\) is a strictly increasing function on \((-\infty, \hat{x}_2)\). By a simple \(v\cdot v'\) phase plane analysis, we conclude that \(\hat{x}_2 < \infty\) and \(v(\hat{x}_2, 0) = 0\). Hence, \(0 \in A\). This completes the proof of the lemma. \(\square\)

### 3.2 The set \(B\) is open

We shall prove this technically challenge result through a sequence of lemmas.

**Lemma 3.3.** Suppose \(v'(z) > 0\), there exists \(b > z\) such that \(v' > 0\) in \((z, b)\), \(v'(b) = 0\), and \(v''(b) < 0\). If \(b\) is a point of local maximum of \(v\), then \(v'(b) = 0 > v''(b)\), \(\sigma(x) := u(x)v^{n-1}(x)\) satisfies \(\sigma'(b) > 1\), and

\[
v(x) < v(b) \quad \forall x \in (b, \infty).
\]

**Proof.** Let \(b = \sup\{x > z \mid v' > 0 \text{ in } (z, x)\}\). It is easy to show that \(b < \infty\). Thus, \(v'(b) = 0\) and \(v''(b) < 0\).

Now let \([b, a)\) be the maximum interval on which \(v' \leq 0\).

(i) If \(a = \infty\), then \(v' \leq 0\) in \([b, \infty)\) and the assertion of the lemma holds.

(ii) If \(a < \infty\), then we must have \(v'(a) = 0\) and by definition of \(a\), \(v''(a) > 0\). Let \((a, \tilde{b})\) be the maximum interval on which \(v' > 0\). Then \(v' \leq 0\) in \((b, a)\) and \(v' > 0\) in \((a, \tilde{b})\). Hence, \(b\) and \(\tilde{b}\) are neighboring points of local maximum of \(v\).

Note that \(|u(a) - u|v' \leq 0\) in \((b, \tilde{b})\). Hence, for any \(x \in (b, \tilde{b})\), integrating \(v'[Dv'' + u(a)v^n - v] = [u(a) - u]v^n v' - \sigma v'^2 \leq 0\) over \([b, x]\) we obtain

\[
\frac{Dv'(x)^2}{2} + \int_{v(b)}^{v(x)} [u(a)s^{n-1} - 1]ds < 0.
\]

Note that \(u(a)v(b)^{n-1} \geq u(b)v^{n-1}(b) = \sigma(b) > 1\), we then obtain \(v(x) < v(b)\) for every \(x \in (b, \tilde{b})\).

If \(\tilde{b} = \infty\), we obtain the assertion of the lemma. If \(\tilde{b} < \infty\), then \(\tilde{b}\) is a point of local maximum of \(v\) and \(v(\tilde{b}) < v(b)\). Since \(v\) has only finitely many critical points in any finite interval, one can use a mathematical induction to derive the assertion of the lemma. This completes the proof. \(\square\)
Lemma 3.4. Suppose a is a point of local minimum of v. Then

\[ v(x) \leq \left( \frac{n+1}{2u(a)} \right)^{\frac{1}{n-1}} \quad \forall x \geq a. \]

In particular, taking \( a = -\infty \) we have

\[ v(x) \leq \left( \frac{n+1}{2u_0} \right)^{\frac{1}{n-1}} \quad \forall x \in \mathbb{R}. \]

Proof. Let \((a, b)\) be the maximal interval on which \(v\) is strictly increasing. Then \(v'(a) = 0\) and \(v' > 0\) on \((a, b)\). Since \(u' > 0\) on \(\mathbb{R}\), we can integrate \(v'v = v'[Dv'' + cv' + uv^n]\) over \((a, x)\) with \(x \in (a, b)\) to obtain

\[
\frac{v^2(x) - v^2(a)}{2} = \frac{Dv^2(x)}{2} + c \int_a^x v'^2 + \int_a^x v^n v' \, dx > u(a) \int_a^x v^n v' \, dx = \frac{u(a)}{n+1} (v^{n+1}(x) - v^{n+1}(a)).
\]

Thus

\[
\frac{n+1}{2u(a)} > \frac{v^{n+1}(x) - v^{n+1}(a)}{v^2(x) - v^2(a)} = \frac{v^{n-1}(x)}{1 - t^2} \bigg|_{t = \frac{v(d)}{v(x)}} \geq v^{n-1}(x).
\]

In view of Lemma 3.3, we then obtain the assertion of the Lemma. \(\square\)

Lemma 3.5. Let \(\lambda = \frac{2}{c + \sqrt{c^2 + 4D}}\) and \(\alpha = \frac{1}{n-1}\). Then for every \(z \in \mathbb{R}\),

\[ v(x) \leq \frac{1}{c\lambda} \max \left\{ v(z), \frac{1}{u^\alpha(z)} \right\} \quad \forall x \geq z. \]

Proof. Fix \(z \in \mathbb{R}\).

1. First we consider the case \(v'(z) > 0\). Let \((a, b) \supseteq z\) be the maximum interval on which \(v' > 0\). Then \(b < \infty, v'(a) = v'(b), \) and \(\sigma(a) < 1 < \sigma(b)\). In addition, since \(u' > 0\) and \(v' > 0\) in \((a, b)\), we have \(\sigma' > 0\) in \((a, b)\), so there exists \(\bar{z} \in (a, b)\) such that \(\sigma' = 1\). Consider two cases.

(i) Suppose \(z \geq \bar{z}\). Then for \(x \in [z, b]\), we have \(Dv'' + cv' = v[1 - \sigma] \leq 0\) in \([z, x]\). After integration, we obtain,

\[ v(x) \leq \frac{-Dv'(x) + Dv'(z) + cv(z)}{c} \leq \frac{\lambda Dv(z) + cv(z)}{c} \leq \frac{v(z)}{c\lambda}, \]

since \(v'(x) \geq 0, v'(z) < \lambda v(z)\) (from Lemma 3.1), and \(\lambda(D\lambda + c) = 1\).

(ii) Suppose \(z < \bar{z}\). Then \(v(x) < v(\bar{z})\) for \(x \in [z, \bar{z}]\) and \(v(x) \leq v(\bar{z})/(c\lambda)\) when \(x \in [\bar{z}, b]\). Hence, for \(x \in [z, b]\),

\[ v(x) \leq \frac{v(\bar{z})}{c\lambda} = \frac{1}{c\lambda u^\alpha(\bar{z})} \leq \frac{1}{c\lambda u^\alpha(z)}. \]
Lemma 3.7. Let \( \eta \) be defined as
\[
\eta = \left\{ \begin{array}{ll}
\eta(x) &= \frac{(c\lambda)^{-1}}{u^{-a}(z)} \max \{v(z), u^{-a}(z)\} \\
& \text{for every } x \in [z, b].
\end{array} \right.
\]
After applying Lemma 3.3, we obtain the assertion of the lemma.

2. Next, suppose \( v'(z) \leq 0 \). Let \([z, \alpha]\) be the maximum interval on which \( v' \leq 0 \). In the case \( a = \infty \), we have \( v(x) \leq v(z) \) for all \( x \in [z, \alpha] = [z, \infty) \), so the assertion of the lemma is true.

If \( a < \infty \), then \( v(x) \leq v(z) \) for every \( x \in [z, a] \). Using the first step, we have, for every \( x > a \), \( v(x) \leq (c\lambda)^{-1} \max \{v(a), u^{-a}(a)\} \leq (c\lambda)^{-1} \max \{v(z), u^{-a}(z)\} \). This completes the proof of the lemma.

Lemma 3.6. \( \lim_{x \to x^+} v(x) = 0 \). Let \( \rho = u'/u \). Then for every \( z \in \mathbb{R} \) and \( h \geq 0 \),
\[
\rho(z + h) \leq \frac{e^{-ch}}{c} \left( \frac{n+1}{2u_0} \right)^{\frac{1}{c}} + \frac{1-e^{-ch}}{c} \sup_{[z, \infty)} v^n',
\]
(3.5)
\[
\rho(z + h) \leq \max \left\{ \rho(z), \frac{1}{c} \sup_{[z, \infty)} v^n' \right\}.
\]
Consequently, \( \lim_{x \to \infty} \rho(x) = 0 \)

Consequently, \( \lim_{x \to \infty} \rho(x) = 0 \) and
\[
0 < \rho(x) \leq \frac{1}{c} \left( \frac{n+1}{2u_0} \right)^{\frac{1}{c}} \quad \forall x \in \mathbb{R}.
\]
(3.7)
The proof is straight forward and can be found in [8, 24].

Consider the function \( w = u^a v \). We have
\[
Dw'' + cw' = u^a [Dv'' + cv'] + 2D(u^a) v' + \alpha u^{-1} [Du'' + D(\alpha-1)u^{-1}u^2 + cu'] v
\]
\[
= w - w^n + \frac{2D(u^a)^'}{u^a} \left( [u^a v'] - [u^a] v' \right) + \frac{\alpha w}{u} \left\{ Duv^n_+ + (1-D)cu' + D(\alpha-1) \frac{u^2}{u} \right\}.
\]
Hence,
\[
Dw'' + cw' - w + w^n_n = \eta_1 w' + \eta_2 w,
\]
(3.8)
where
\[
\eta_1 = 2\alpha D\rho, \quad \eta_2 = \alpha D\rho_+ + \alpha [1-D]c\rho - D\alpha [\alpha + 1] \rho^2.
\]
(3.9)
Define \( \eta(x) = \max_{z>x} [||\eta_1(z)|| + ||\eta_2(z)||] \). Then by Lemma 3.6 we have
\[
\lim_{x \to \infty} \eta(x) = 0.
\]
(3.10)

Lemma 3.7. Suppose that \( x_2 = \infty \). Then one of the following holds:

1. \( \lim_{x \to \infty} w(x) = 1 \) and \( \lim_{x \to \infty} u(x) = \infty \), so \( c \in B \);
(2) \( \lim_{x \to \infty} w(x) = 0 \) and \( \lim_{x \to \infty} u(x) < \infty \), so \( c \in \mathcal{C} \). In addition,

\[
\lim_{x \to \infty} \frac{w'(x)}{v(x)} = -\mu = -\frac{c + \sqrt{c^2 + 4D}}{2D}.
\]

(3.11)

**Proof.** One and only one of the following holds:

1. \( \liminf_{x \to \infty} w^{n-1}(x) > 1 + \frac{c^2}{4D} \);
2. \( \liminf_{x \to \infty} w^{n-1}(x) \leq 1 + \frac{c^2}{4D} \) and \( \limsup_{x \to \infty} w(x) > \liminf_{x \to \infty} w(x) \);
3. \( \liminf_{x \to \infty} w^{n-1}(x) = \limsup_{x \to \infty} w^{n-1}(x) \leq 1 + \frac{c^2}{4D} \).

We shall show that only case (3) happens.

1. Suppose \( \liminf_{x \to \infty} w^{m-1}(x) > 1 + \frac{c^2}{4D} \). Consider the function \( \hat{\vartheta}(x) = v(x)e^x/2D \). Then \( v = e^{-\alpha x}/2D \hat{\vartheta} \), and

\[
D\hat{\vartheta}'' = \hat{\vartheta}Q, \quad Q := 1 + \frac{c^2}{4D} - w^{n-1}.
\]

Note that \( \liminf_{x \to \infty} Q(x) < 0 \). It then follows from the Sturm–Liouville nodal comparison theorem that \( \hat{\vartheta} \) oscillates, i.e., \( \hat{\vartheta} \) changes sign infinitely many times as \( x \to \infty \), contradicting the assumption that \( x_2 = \infty \). Hence, case (1) does not happen.

2. Suppose \( l := \liminf_{x \to \infty} w^{m-1}(x) \leq 1 + \frac{c^2}{4D} \) and \( L := \limsup_{x \to \infty} w^{n-1}(x) > l \). Then \( w \) oscillates infinitely many times as \( x \to \infty \).

Let \( \{b_j\}_{j=1}^{\infty} \) be a sequence of point of local maximum of \( w \) such that \( b_j \to \infty \) and \( w^{n-1}(b_j) \to L \) as \( j \to \infty \). By (3.10), we can assume that \( \eta(b_1) < \min(1, c) \). For each \( j \geq 2 \), let \( (a_j, b_j) \) be the largest interval such that \( w' > 0 \) in \( (a_j, b_j) \). Then \( w'(a_j) = w'(b_j) = 0 \). Integrating \( Dw''w'' + w''w' = [-c + \eta_1]w'^2 + [1 + \eta_2]ww' \leq [1 + \eta(a_j)]ww' \) over \( [a_j, b_j] \) we obtain

\[
\frac{w^{n+1}(b_j) - w^{n+1}(a_j)}{n+1} \leq \frac{1 + \eta(a_j)}{2} [w(b_j)]^2 - w^2(a_j).
\]

This implies that

\[
w^{n-1}(b_j) \leq \frac{(n+1)[1 + \eta(a_j)]}{2} \sup_{0 < t < 1} \frac{1 - t^2}{1 + \mu + 1} \leq \frac{(n+1)[1 + \eta(a_j)]}{2}.
\]

Sending \( j \to \infty \) we find that \( L \leq \frac{n+1}{2} \). Now set \( w_j(x) = w(b_j + x) \). Then

\[
\lim_{j \to \infty} \left( w_j(0), w_j'(0) \right) = (L^\alpha, 0).
\]

Hence, from (3.8) and (3.10), we see that \( w_* := \lim_{j \to \infty} w_j \) exists and satisfies

\[
Dw_*'' + cw_*' + w_* - w_* = 0 \quad \text{in} \quad \mathbb{R}, \quad w_*'(0) = 0, \quad w_*(0) = L^\alpha = \sup_{x \in \mathbb{R}} w_*(x) > 0. \quad (3.12)
\]
Since \( c > 0 \) and \( w_* \geq 0 \), the only solution is \( w_* = 1 \). Hence, \( L = 1 \).

Similarly, let \( \{ \hat{a}_j \}_{j=1}^{\infty} \) be a sequence of local minimum of \( w \) such that as \( j \to \infty, \hat{a}_j \to \infty \) and \( w(\hat{a}_j) \to l \). For each \( j \geq 2 \), let \( (\hat{b}_j, \hat{a}_j) \) be the maximum interval on which \( w' < 0 \). First of all, we derive from \( w'(\hat{b}_j) = 0 \geq w''(\hat{b}_j) \) that

\[
0 \geq Dw''(\hat{b}_j) = w(\hat{b}_j)[1 - w^{n-1}(\hat{b}_j) + \eta_2(\hat{b}_j)],
\]

so \( w(\hat{b}_j) \geq [1 + \eta_2(\hat{b}_j)]^{1/n} \). As \( \limsup_{x \to \infty} w(x) = L = 1 \), we find that \( \lim_{x \to \infty} w(\hat{b}_j) = 1 \). Next, integrating \( w'[ Dw'' + w^n] = [-c + \eta_1]w^2 + [1 + \eta_2]ww' \leq [1 - \eta(\hat{b}_j)]w' \) over \( [\hat{b}_j, \hat{a}_j] \) we derive that

\[
\frac{w^{n+1}(\hat{a}_j) - w^{n+1}(\hat{b}_j)}{n+1} \leq \frac{[1 - \eta(\hat{b}_j)][w(\hat{a}_j)^2 - w^2(\hat{b}_j)]}{2}.
\]

Sending \( j \to \infty \) we derive that

\[
\frac{\int_{(n+1)\alpha}^{2\alpha} - \int_{0}^{2\alpha} 1}{n+1} \leq \frac{1}{n+1} - \frac{1}{2}.
\]

This implies that \( L = 1 \). Hence, \( \lim_{x \to \infty} w(x) = 1 \). But this contradicts \( \ell < L \). Hence case (2) does not happen.

(3) Suppose \( L := \lim_{x \to \infty} w^{n-1}(x) \) exists and is finite. Then we derive from (3.8) and (3.10) that either \( L = 0 \) or \( L = 1 \).

If \( L = 0 \), then we derive from

\[
Dv'' + cv' = v[1 - w^{n-1}] = [1 + o(1)]v
\]

that \( v \) decays to zero exponentially fast as \( x \to \infty \). Consequently, both \( \int_{\mathbb{R}} vdx \) and \( \int_{\mathbb{R}} uv''dx = \int_{\mathbb{R}} w^{n-1}v \) are finite. Using integrating factor \( e^{cx} \) we then obtain, for every \( x \in \mathbb{R} \),

\[
u(x) = u_0 + \int_{-\infty}^{x} e^{c(y-x)} \int_{-\infty}^{y} u(z)v''(z)dz dy \leq u_0 + \frac{1}{c} \int_{\mathbb{R}} u(z)v''(z)dz.
\]

Hence, \( \lim_{x \to \infty} u(x) < \infty \) so \( c \in C \). In addition, writing the equation for \( v \) as \( Dw'' + cv' - v = O(v^n) \) we obtain (3.11). This completes the proof of the Lemma.

The key step to show \( B \) is open is to construct an Invariant Region. Let

\[
g(s) = s^n - s, \quad G(s) = \int_{0}^{s} g(t)dt, \quad \hat{g}(s) = \begin{cases} g(s) & \text{if } s \in [0, 2], \\ g(2) + g'(2)(s-2) & \text{if } s > 2, \end{cases}
\]

\[
k_1 = \inf_{0 < s \neq 1} \frac{DG(s)}{2g^2(s)}, \quad k_2 := \inf_{1 < s < 2, s \neq 1} \frac{g^2(s)}{G(s)}, \quad \theta = \min \left\{ \sqrt{k_1}, \frac{c}{4g'(2)}, \frac{D}{4c} \right\}.
\]
Since $g$ is convex and $g'(1) = 0 < g'(1)$, $k_1$ and $k_2$ are positive and in $[0, \infty)$, 
\[ g \geq \hat{g}, \quad G \geq \frac{2\theta^2}{D} \hat{g}^2, \quad \theta \hat{g}' \leq \frac{c}{4}, \quad \hat{g} \left[ g - \frac{2c\theta}{D} \hat{g} \right] \geq \frac{1}{2} \hat{g} \hat{s} \geq \frac{1}{2} s^2. \]

Consider the “energy”:
\[ E := \frac{D}{2} w'^2 + G(w) + \theta \hat{g}(w)w'. \]  
(3.13)

By the definition of $\theta$, we have $|\theta \hat{g}(w)w'| \leq \frac{D}{4} w'^2 + \frac{\hat{g}^2}{2} \leq \frac{D}{4} w'^2 + \frac{1}{2} G$. Hence,
\[ \frac{3}{4} D w'^2 + \frac{3}{2} G(w) \geq E \geq \frac{D}{4} w'^2 + \frac{1}{2} G(w). \]  
(3.14)

We can calculate, when $x < x_2$,
\[
E' = w'[Dw'' + g(w)] + \theta \hat{g}(w)w'' + \theta \hat{g}'(w)w'^2
\leq [\eta_1 - c + \theta \hat{g}']w'^2 - \frac{\theta}{D} \hat{g} \hat{s} \hat{g}^2 + \left( \frac{c w'^2}{4} + \eta_2^2 w^2 \right) +
\left( \frac{c}{4} \theta \hat{g}^2 + \frac{\eta_2^2 w^2}{3c} \right)
\leq -\left( \frac{c}{4} - \frac{\eta_1^2}{c} \right) w'^2 - \frac{\theta}{2D} \hat{g} \hat{s} \hat{g}^2 + \frac{4\eta_2^2 w^2}{3c}.
\]

In particular, if $0 < \eta_1 < \frac{2}{3}$ and $|w - 1| \leq \frac{1}{2}$ we have
\[
E' \leq -\frac{c}{8} w'^2 - \frac{\theta k_2}{2D} G(w) + \frac{3\eta_2^2}{c}
\leq -\min \left\{ \frac{c}{6D} + \frac{\theta k_2}{3D} \right\} E + \frac{3\eta_2^2}{c}. \]  
(3.15)

**Lemma 3.8.** Suppose $c > 0$ and there exists $z < x_2$ such that, with $\delta = \delta(c)$ defined in (3.17) below,
\[
\rho(z) \leq \frac{\delta}{c}, \quad v(z) \leq (c\lambda) \delta^+ \leq \frac{1}{u^\alpha(z)} \leq (c\lambda) \delta^+, \quad E(z) \leq \frac{1}{2} G \left( \frac{1}{2} \right), \]  
(3.16)

then $x_2 = \infty$, $\lim_{x \to \infty} w(x) = 1$ and $\lim_{x \to \infty} u(x) = \infty$.

**Proof.** First of all, by Lemma 3.5,
\[
\sup_{[z, \infty]} v \leq \frac{1}{c\lambda} \max \left\{ v(z), \frac{1}{u^\alpha(z)} \right\} \leq \delta^+. \]  
(3.17)
Next, by (3.6),

$$\sup_{[z,\infty)} \rho \leq \max \left\{ \rho(z), \frac{1}{c} \sup_{[z,\infty)} v'' \right\} \leq \frac{\delta}{c}. $$

Assume that $\delta \leq c^2$. Then $\rho^2 \leq \delta^2/c^2 \leq \delta$. It then follows from the definition of $\eta_i$ in (3.9) that in $[z,\infty)$,

$$|\eta_1| \leq \frac{2\alpha D \delta}{c}, \quad |\eta_2| < \alpha \max \{D,1\}[2+\alpha]\delta. $$

Now we define

$$\delta(c) = \min \left\{ c^2, \frac{c^2}{18\alpha D}, \frac{1}{\alpha \max \{D,1\}[2+\alpha]} \left( \frac{G(\frac{1}{2})}{6} \min \left\{ \frac{c^2}{6D}, \frac{c\theta k_2}{3D} \right\} \right)^{\frac{1}{2}} \right\}. $$

Then in $[z,\infty)$ we have

$$|\eta_1| \leq \frac{\alpha}{9}, \quad -\min \left\{ \frac{c}{6D}, \frac{\theta k_2}{3D} \right\} \frac{1}{2} G \left( \frac{1}{2} \right) + \frac{3\eta_2^2}{c} < 0.$$

Now set

$$\hat{x}_2 = \sup \left\{ x \in [z,x_2) \mid 2E(x) \leq G \left( \frac{1}{2} \right) \right\}.$$

We claim that $\hat{x}_2 = x_2$.

Suppose the claim is not true. Then $\hat{x}_2 < x_2$. Consequently, at $\hat{x}_2$, we have $2E(\hat{x}_2) = G(\frac{1}{2})$ and $E'(\hat{x}_2) \geq 0$. That $2E(\hat{x}_2) \leq G(1/2)$ implies that $G(w(\hat{x}_2)) \leq G(1/2)$ so $|w(\hat{x}_2) - 1| \leq 1/2$. Consequently, by (3.15), we have

$$E'(x_2) \leq -\min \left\{ \frac{c}{6D}, \frac{\theta k_2}{3D} \right\} E(x_2) + \frac{3\eta_2^2}{c} \leq -\min \left\{ \frac{c}{6D}, \frac{\theta k_2}{3D} \right\} \frac{1}{2} G \left( \frac{1}{2} \right) + \frac{3\eta_2^2}{c} < 0,$$

a contradiction. Thus, we must have $\hat{x}_2 = x_2$.

Now we claim that $x_2 = \infty$. Indeed, if $x_2 < \infty$, we must have $v(x_2) = 0$ and $v'(x_2) < 0$. This implies that $w(x_2) = 0$ and by (3.14), $\lim_{x \to x_2} 2E(x_2) \geq G(0)$. But this contradicts $2E \leq G(1/2) < G(0)$ on $[z,x_2)$. Thus, we must have $x_2 = \infty$.

Finally, since $\lim_{x \to \infty} \eta_2 = 0$, we derive from (3.15) that $\lim_{x \to \infty} E(x) = 0$. This implies that $\lim_{x \to \infty} w(x) = 1$ and completes the proof of the Lemma.

Now we are ready to show the following:

**Lemma 3.9.** The set $B$ defined in (3.3) is open.
Lemma 3.10. Suppose $c_0 \in \mathcal{B}$. Then we have

$$\lim_{x \to \infty} \left( |\rho(x,c_0)| + |w(x,c_0)| + \frac{1}{|u(x,c_0)|} + |w'(x,c_0)| + |w(x,c_0) - 1| \right) = 0.$$  

Hence, by continuous dependence of solution with respect to parameter, there exists $e > 0$ and $z \in \mathbb{R}$ such that (3.16) holds for every solution of (2.4) with $c \in (c_0 - e, c_0 + e)$. Then by Lemma 3.8, $c \in \mathcal{B}$ for every $c \in (c_0 - e, c_0 + e)$. Hence, $\mathcal{B}$ is open.

Lemma 3.10. Let

$$V_0 := \left( \frac{n+1}{2m_0} \right)^{\frac{1}{p}}, \quad M = \frac{n+1}{2} \left[ 1 + \alpha \max \{ D, 1 \} V_0^n \right], \quad C = \sqrt{4d(M - 1)}.$$  

Then $[C, \infty) \subset \mathcal{B}$.

Proof. Suppose $c \geq C$. We show that $c \in \mathcal{B}$ by three steps.

1. By (3.4) we have $v \leq V_0$. By (3.7) we have $0 < \rho \leq V_0^n / c$. Hence, by the definition of $\eta_1$ and $\eta_2$,

$$0 < \eta_1(x) \leq \frac{2\Delta V_0^n}{c} < c, \quad \eta_2(x) \leq \alpha \max \{ D, 1 \} V_0^n.$$  

2. Next we find an upper bound of $w := u^a v$. Let $\tilde{M} = (\sup_{\mathbb{R}} w)^{n-1}$. Then there exists a sequence $\{ z_i \}_{i=1}^\infty$ in $(-\infty, x_2)$ such that $w'(z_i) > 0$ and $\lim_{i \to \infty} w^{n-1}(z_i) = \tilde{M}$. For each $i \geq 1$, let $(a, z_i)$ be the maximal interval in which $w' > 0$. Then integrating $w^a w^n = - Dw^a w^n - (c - \eta_1) w^2 + (1 + \eta_2) w w' \leq - Dw^a w^n + [1 + \sup_{\mathbb{R}} \eta_2] w w'$ over $(a, z_i)$ we obtain

$$\frac{w^{n+1}(z_i) - w^{n+1}(a)}{n+1} \leq \left[ 1 + \sup_{\mathbb{R}} \eta_2 \right] \frac{w^2(z_i) - w^2(a)}{2}. $$

Thus,

$$w^{n-1}(z_i) \leq \frac{n+1}{2} \left[ 1 + \sup_{\mathbb{R}} \eta_2 \right] \frac{1 - \frac{1}{n+1}}{1 - \frac{1}{n+1}} \left[ 1 + \sup_{\mathbb{R}} \eta_2 \right] \frac{n+1}{2} \left[ 1 + \alpha \max \{ D, 1 \} V_0^n \right] = M.$$  

Sending $i \to \infty$ we obtain $(\sup_{\mathbb{R}} w)^{n-1} \leq M$.

3. We now show that $c \in \mathcal{B}$. Let $\tilde{\mu} = \frac{c + \sqrt{c^2 + 4D(1-M)}}{2D}$. Then $D \tilde{\mu}^2 - c \tilde{\mu} + M - 1 = 0$. In $(-\infty, x_2)$,

$$D[v' + \tilde{\mu} v]' + (c - D \tilde{\mu})[v' + \tilde{\mu} v] = Dv' + cv' + (c \tilde{\mu} - D \tilde{\mu}^2) v = (c \tilde{\mu} - D \tilde{\mu}^2 + 1 - w^{n-1}) v \geq (c \tilde{\mu} - D \tilde{\mu}^2 + 1 - M) v = 0.$$  

This implies that $(e^{c/D - \tilde{\mu} x} [v' + \tilde{\mu} v])' \geq 0$. Since $v' + \tilde{\mu} v > 0$ for $x < x_1$, after integration, we derive that

$$v'(x) + \tilde{\mu} v(x) > 0 \quad \forall x < x_2.$$
Since \( x_2 < \infty \) would imply \( v(x_2) = 0 > v'(x_2) \), we see that \( x_2 = \infty \). In addition,

\[
\liminf_{x \to \infty} \frac{v'(x)}{v(x)} \geq -\hat{\mu} > -\mu.
\]

It then follows from Lemma 3.7 (in particular (3.11)) that we must have \( c \in \mathcal{B} \). This completes the proof of the lemma.

\[ \square \]

### 3.3 Proof of Theorem 1.3

**Proof of Theorem 1.3.** Since \( [0, \infty) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) and \( \mathcal{A} \) and \( \mathcal{B} \) are open and disjoint non-empty sets, \( \mathcal{C} \) is non-empty. When \( c \in \mathcal{C} \), the solution of (2.4) is a solution of (1.4). The bound on the speed follows directly from Lemma 3.10. This completes the proof of Theorem 1.3.

\[ \square \]

### Acknowledgments

The author thanks Min Chen and Yi Li for stimulating discussions.

### References


