Overdetermined Boundary Value Problems in $S^n$

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Received 1 July, 2016; Accepted 7 March, 2017

Abstract. In this paper we use the maximum principle and the Hopf lemma to prove symmetry results to some overdetermined boundary value problems for domains in the hemisphere or star-shaped domains in $S^n$. Our method is based on finding suitable $P$-functions as Weinberger ([26]).

AMS subject classifications: 35J15, 35R01

Key words: Overdetermined problems, Schiffer’s problem, P-function.

1 Introduction

In a seminal paper [21], Serrin proved that for a bounded open connected domain $\Omega \subset \mathbb{R}^n$ with sufficient regular boundary $\partial \Omega$, if there exists a solution of the following overdetermined boundary value problem

$$\begin{cases}
\Delta u = n & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} = c & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $c$ is a constant, then $\Omega$ must be a ball and $u$ is radially symmetric. Here $\nu$ denotes the outward unit normal of $\partial \Omega$.

The main tool of Serrin’s proof is well-known as the method of moving planes, which is due to Alexandrov. Immediately after Serrin’s paper, Weinberger [26] give an alternative proof of the same result, based on a Rellich-Pohozaev type identity and an interior maximum principle for a subharmonic function (In literatures, it is often referred to as P-function). Each of their proofs has its own merits. Serrin’s argument applies to very...
general partial differential equations if an additional assumption $u > 0$ is added, while Weinberger’s argument is more elementary.

Since the works of Serrin and Weinberger, there have been numerous generalizations to overdetermined problems for general elliptic operators in $\mathbb{R}^n$, the interested readers may refer to [4–6, 8, 11–14, 17, 25] and references therein.

On the other hand, Serrin’s result has been extended to the hemisphere $S^n_+$ and the hyperbolic space $H^n$. Precisely, Molzon [16] considered equation $\Delta u = f(x)$ where $f(x) = \cos r (\cosh r \text{ resp.})$ in the case $S^n_+$ ($H^n$ resp.) and $r$ is the distance function from a fixed point or $f(x) = n$. Kumaresan and Prajapat [15] considered equation $\Delta u + f(u) = 0$ in $\Omega \subset S^n_+$ or $H^n$, where $f$ is a $C^1$ function. They proved that if $\Delta u + f(u) = 0$ with the boundary condition $u = 0$ and $\frac{\partial u}{\partial \nu} = \text{constant}$ admits a positive solution, then $\Omega$ is a geodesic ball and $u$ is radially symmetric. They used Serrin’s method of moving planes to achieve this, where the positivity of $u$ is an unremovable assumption.

In this paper, we will study an overdetermined problem corresponding to a particular equation on $S^n$:

$$\begin{align*}
\Delta u + nu &= n \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} &= c \quad \text{on } \partial \Omega.
\end{align*}$$

Equation (1.2) is related to Schiffer’s problem (See [28] problem 80) on $S^n$. See for instance [1–3, 7, 9, 10, 23, 24, 27] for recent developments of Schiffer’s problem. Previously, Souam [24] showed that for $n = 2$ and $\Omega \subset S^n$ simply connected, if (1.2) admits a solution, then $\Omega$ must be a geodesic ball.

Our first result is the following.

**Theorem 1.1.** Let $\Omega \subset S^n$ be a bounded open connected domain such that $\overline{\Omega}$ is contained in a hemisphere $S^n_+$. If the overdetermined problem (1.2) admits a solution $u$, then $\Omega$ must be a geodesic ball and $u$ is radially symmetric.

We remark that since the first Dirichlet eigenvalue for a domain $\overline{\Omega} \subset S^n_+$ is strictly larger than $n$, there exists a unique solution for the Dirichlet problem $\Delta u + nu = n$ in $\Omega$ and $u = 0$ on $\partial \Omega$. However, it is not a priori known whether the solution has a definite sign. Therefore, Theorem 1.1 does not follow from the result of Kumaresan and Prajapat [15].

Our approach to Theorem 1.1 is parallel to Weinberger’s, namely, we use a maximum principle for a subharmonic function $P$ and a Rellich-Pohozaev type identity. We remark that our method also applies to equation $\Delta u - nu = n$ in $\Omega \subset H^n$. In this case, $u$ is negative in $\Omega$ by the maximum principle. Hence the conclusion also follows from the result of Kumaresan and Prajapat.

Our next result concerns the same overdetermined problem (1.2) in $\Omega \subset S^n$ without the assumption that $\overline{\Omega}$ is contained in a hemisphere $S^n_+$. Instead, we shall add a star-shapedness assumption on $\Omega$. A domain $\Omega \subset S^n$ is called star-shaped with respect to $p \in S^n$ if $\Omega$ can be written as a graph over a geodesic sphere centered at $p$. It is clear that a
domain $\Omega$ which is star-shaped w.r.t. $p$ does not contain the antipodal point $-p$ and the unique geodesic connecting $p$ and $q \in \Omega$ is contained in $\Omega$.

**Theorem 1.2.** Let $\Omega$ be a bounded open connected domain in $S^n$. Assume that $\Omega$ is star-shaped with respect to some fixed point $p \in S^n$. If the overdetermined problem (1.2) admits a solution $u$, then $\Omega$ must be a geodesic ball and $u$ is radially symmetric.

For the proof of Theorem 1.2, we find another harmonic function $\tilde{P}$. By examining the boundary behavior of $P$ and $\tilde{P}$ and using the Hopf lemma, we see either $P$ or $\tilde{P}$ is a constant function, which implies our conclusion.

We remark that, in general one can construct round symmetric annuli on $S^n$ such that (1.2) has solutions.† That means, the condition that $\Omega$ is contained in a hemisphere or $\Omega$ is star-shaped cannot be totally removed.

**Notation:** In the following sections, we denote by $\Delta$ and $\nabla$ the Laplacian and the gradient on $S^n$ respectively. We denote by $d\Omega$ and $dA$ the volume measure and the surface measure of $\Omega$ and $\partial \Omega$ respectively. For simplicity, we will use $u, u_{ij}, \cdots$ and $u_v$ to denote covariant derivatives and normal derivative of a function $u$ with respect to the round metric on $S^n$ respectively. We will also follow Einstein’s summation convention.

## 2 Two $P$-functions

In this section we find two $P$-functions $P$ and $\tilde{P}$ for $u$ satisfying equation $\Delta u + nu = n$ on $S^n$. The first $P$-function in Lemma 2.1 is analogous to the one in $\mathbb{R}^n$ used in [26, 18].

**Lemma 2.1.** Let $u$ satisfies the equation $\Delta u + nu = n$ in $\Omega \subset S^n$. Then the following $P$-function

$$ P := |\nabla u|^2 + u^2 - 2u, \quad (2.1) $$

satisfies subharmonic property

$$ \Delta P \geq 0. \quad (2.2) $$

**Proof.** Let $g$ be the round metric on $S^n$. Since $Ric_g = (n-1)g$, the Bochner formula gives

$$ \Delta |\nabla u|^2 = 2|\nabla^2 u|^2 + 2(\nabla \Delta u, \nabla u) + 2Ric(\nabla u, \nabla u) $$

$$ = 2|\nabla u + ug|^2 - 4u\Delta u - 2nu^2 + 2(\nabla (n - nu), \nabla u) + 2(n - 1)|\nabla u|^2 $$

$$ \geq \frac{2}{n} (\Delta u + nu)^2 - 2|\nabla u|^2 + 2nu^2 - 4nu $$

$$ = 2n - 2|\nabla u|^2 + 2nu^2 - 4nu. \quad (2.3) $$

†We thank Dr. Guanghao Hong for pointing out this to us.
We have used the Schwarz’s inequality \(|\nabla^2 u + ug|^2 \geq \frac{1}{n}(\Delta u + nu)^2\). Direct computation gives

\[
\Delta u^2 = 2|\nabla u|^2 + 2u\Delta u = 2|\nabla u|^2 + 2nu - 2nu^2. \tag{2.4}
\]

Combining (2.3) and (2.4), we have

\[
\Delta P = \Delta |\nabla u|^2 + \Delta u^2 - 2\Delta u \geq 0.
\]

The completes is the proof. \(\Box\)

The second \(P\)-function is a harmonic function.

**Lemma 2.2.** Let \(u\) satisfies the equation \(\Delta u + nu = n\) in \(\Omega \subset S^n\). Let \(p \in \Omega\) and its antipodal point \(-p \notin \Omega\). Denote \(V(x) = \cos(r(x))\), where \(r\) is the distance function from \(p\). Then \(V\) is differentiable in \(\Omega\) and the following \(P\)-function

\[
\tilde{P} := \langle \nabla u, \nabla V \rangle + uV - V, \tag{2.5}
\]

satisfies

\[
\Delta \tilde{P} = 0. \tag{2.6}
\]

**Proof.** Since \(\{-p\}\) is the cut locus of \(p\) and \(-p \notin \Omega\), \(V\) is differentiable in \(\Omega\). It is well known that \(\nabla^2 V = -Vg\) and \(\Delta V = -nV\) in \(\Omega \subset S^n\). Direct computation yields:

\[
\Delta (\nabla V, \nabla u) = \langle \Delta \nabla V, \nabla u \rangle + \langle \nabla V, \Delta \nabla u \rangle + 2\langle \nabla^2 V, \nabla^2 u \rangle
\]

\[
= \langle \nabla \Delta V, \nabla u \rangle + Ric(\nabla V, \nabla u) + \langle \nabla V, \Delta \nabla u \rangle + Ric(\nabla V, \nabla u) - 2V\Delta u
\]

\[
= -n \langle \nabla V, \nabla u \rangle + 2(n-1)\langle \nabla V, \nabla u \rangle + \langle \nabla V, \nabla (n-nu) \rangle - 2V(n-nu)
\]

\[
= -2\langle \nabla V, \nabla u \rangle - 2nV + 2nVu, \tag{2.7}
\]

and

\[
\Delta (Vu) = \Delta Vu + V\Delta u + 2\langle \nabla V, \nabla u \rangle
\]

\[
= -2nVu + 2\langle \nabla V, \nabla u \rangle + nV. \tag{2.8}
\]

Combining (2.7) and (2.8), we conclude \(\Delta \tilde{P} = -Vn - \Delta V = 0\). \(\Box\)

3 A Rellich-Pohozaev type identity

In order to prove Theorem 1.1, we need the following Rellich-Pohozaev type identity.
Lemma 3.1. Let $u$ be a solution of problem (1.2) and $V(x)$ defined as in Lemma 2.2. Then the following integral identity holds:

$$\int_{\Omega} \left( -\frac{n(n+2)}{2} Vu^2 + (n+2)n Vu \right) d\Omega = \int_{\partial \Omega} c^2 V \nu \, dA.$$  \hfill (3.1)

Proof. Multiplying $\Delta u + nu$ by $\langle \nabla u, \nabla V \rangle$ and integrating among $\Omega$, we have from integration by parts that

$$\int_{\Omega} (\Delta u + nu) \langle \nabla u, \nabla V \rangle d\Omega = \int_{\Omega} n \langle \nabla u, \nabla V \rangle d\Omega = \int_{\Omega} n^2 u V d\Omega.$$  \hfill (3.2)

Here we have used $V_{ij} = -V_{g_{ij}}$ and $u = 0$ on $\partial \Omega$. On the other hand, since $\Delta u + nu = n$,

$$\int_{\Omega} (\Delta u + nu) \langle \nabla u, \nabla V \rangle d\Omega = \int_{\Omega} n \langle \nabla u, \nabla V \rangle d\Omega.$$  \hfill (3.3)

Combine (3.2) and (3.3), we obtain (3.1).

4 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $V(x) = \cos(r(x))$, where $r$ is the distance function from the origin of $S^n$. It follows from Lemma 2.1 and the strong maximum principle that $P \leq c^2$ in $\Omega$ and only two cases happen:

Case 1: $P \equiv \text{constant}$;

Case 2: $\exists$ a point $x \in \Omega$, such that $P < c^2$.

In the first case, $\Delta P \equiv 0$, then equality must hold in (2.3). It follows that $\nabla^2 u + ug$ $\equiv \lambda(x)g$ for some function $\lambda$. Taking into account of $\Delta u + nu = n$, we see $\nabla^2 u + ug$ $\equiv g$. 

\[ \]
Hence $\nabla^2 (u - 1) = -(u - 1) g$ and $u - 1|_{\partial \Omega} = -1$. It follows from an Obata type result (see Reilly [20]) that $\Omega$ must be a geodesic ball and $u$ must be radial symmetric, precisely, $u(x) = -\sqrt{1+c^2} \cos r(x) + 1$, and $\Omega$ is a geodesic ball of radial $\arccos \left( \frac{1}{\sqrt{1+c^2}} \right)$.

We will show that the second case cannot happen. Suppose it happens. Then thanks to the smoothness of $P$, by integrating $VP$ over $\Omega$, we obtain

$$\int_\Omega |V| \nabla u|^2 + Vu^2 - 2Vu \, d\Omega < c^2 \int_\Omega V \, d\Omega \tag{4.1}$$

note here we use $V > 0$ on $S^n_+$. Integrating by part, we have

$$\int_\Omega |\nabla u|^2 V \, d\Omega = \int_\Omega (\nabla u \cdot \nabla V - u \Delta u V) \, d\Omega + \int_{\partial \Omega} uu_v V \, dA$$

$$= \int_\Omega \left( -\frac{u^2 n V}{2} - u \Delta u V \right) \, d\Omega$$

$$= \int_\Omega \left( -nu V + \frac{nu^2 V}{2} \right) \, d\Omega. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$(n + 2) \int_\Omega \left( -u V + \frac{1}{2} u^2 V \right) \, d\Omega < c^2 \int_\Omega V \, d\Omega. \tag{4.3}$$

On the other hand, Lemma 3.1 tells us that

$$\int_\Omega \left( -\frac{(n+2)n}{2} Vu^2 + (n+2)n Vu \right) \, d\Omega = \int_{\partial \Omega} c^2 V_\nu dA = -\int_\Omega c^2 n V \, d\Omega. \tag{4.4}$$

We see (4.4) contradicts with (4.3). We complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $\Omega \subset S^n$ be a star-shaped domain with respect to $p$ and $V(x) = \cos(r(x))$, where $r$ is the distance function from $p$. We claim that $P \equiv constant$ or $\bar{P} \equiv constant$ in $\Omega$.

Suppose not, i.e., neither $P$ nor $\bar{P}$ is a constant. On one hand, by applying the Hopf lemma to $P$, we know $P_\nu > 0$ on the whole boundary $\partial \Omega$ because $P = c^2$ on $\partial \Omega$. Thus for every point on $\partial \Omega$,

$$0 < P_\nu = 2 \sum_{i=1}^n u_i u_{\nu_i} + 2 uu_\nu - 2u_\nu = 2u_\nu (u_{\nu\nu} - 1). \tag{4.5}$$

Here we used $u = 0$ and $u_\nu = c$ on $\partial \Omega$. Hence

$$c(u_{\nu\nu} - 1) > 0 \quad \text{on} \quad \partial \Omega. \tag{4.6}$$
On the other hand, by applying the Hopf lemma to $\tilde{P}$, at the maximum point of $\tilde{P}$, say $y_1 \in \partial \Omega$, we have $\tilde{P}_v(y_1) > 0$. Thus at $y_1$,

$$0 < \tilde{P}_v(y_1) = \sum_{i=1}^{n} u_{iv} V_i + u_i V + u V_v - V_v
= u_{vv} V_v - u_v V + u_v V - V_v
= u_{vv} V_v - V_v. \quad (4.7)$$

If $\Omega$ is star-shaped, then $V_v = -\sin r \langle \partial_r, v \rangle < 0$ on $\partial \Omega$. So we deduce from (4.7) that

$$u_{vv}(y_1) < 1. \quad (4.8)$$

Similarly, at the minimum point of $\tilde{P}$, say $y_2 \in \partial \Omega$, we have

$$0 > \tilde{P}_v(y_2) = u_{vv} V_v - V_v.$$

So we get

$$u_{vv}(y_2) > 1. \quad (4.9)$$

One of (4.8) and (4.9) must contradict with (4.6), because $c$ is a constant. Therefore, we have shown that only two cases happen:

**Case 1**: $P \equiv \text{constant},$

**Case 2**: $\tilde{P} \equiv \text{constant}.$

In the first case, $\Delta P \equiv 0$, then the conclusion follows from the same argument as the proof of Theorem 1.1. In the second case, $\tilde{P}_v = 0$ on the boundary which implies that $u_{vv} = 1$. Then $P_v = c(u_{vv} - 1) = 0$ and by the Hopf lemma, $P$ must be a constant and we reduce to the first case. We complete the proof of Theorem 1.2.

**Acknowledgments**

The research of the first author is supported in part by NSFC (11371336). The research of the second author is supported in part by NSFC (11501480) and the Fundamental Research Funds for the Central Universities (20720150012). Both authors would like to express their gratitude to Prof. Xinan Ma and Prof. Guofang Wang for their constant encouragement and support.

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